

2.1 Consider the following classical trajectory of an open string: $X^0 = \tau$, $X^1 = B \cos \tau \cos \sigma$, $X^2 = B \sin \tau \cos \sigma$, $X^3 = 0$ i>2.

Assume conformal gauge.

(i) Show that this is a solution to the eom. w/ Neumann boundary conditions.

Show that the ends of the string move at the speed of light.

$$\text{Sigma model action: } S_\sigma = -\frac{1}{2} T \int d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \xrightarrow{h_{\alpha\beta} = \eta_{\alpha\beta}} +\frac{1}{2} T \int d\sigma (\partial_\alpha X^\mu \partial_\alpha X_\mu)$$

$$\text{Variation: } \delta S_\sigma = T \int d\sigma \partial_\alpha X^\mu \partial^\alpha \delta X_\mu$$

$$= T \int d\sigma \partial^\alpha (\partial_\alpha X^\mu \delta X_\mu) - \partial^\alpha \partial_\alpha X^\mu \delta X_\mu$$

$$\rightarrow \text{eom: } \partial^\alpha \partial_\alpha X^\mu = -\ddot{X}^\mu + X^{\mu\prime\prime} = 0$$

$$\rightarrow \text{BC: } T \int d\sigma d\tau \partial^\tau (\dot{X}^\mu \delta X_\mu) + \partial^\sigma (X^{\mu\prime} \delta X_\mu) = 0$$

$$\begin{aligned} T \int d\sigma [\dot{X}^\mu \delta X_\mu]_{\tau_1}^{\tau_2} + T \int d\tau [X^{\mu\prime} \delta X_\mu]_{\sigma=0}^{\sigma=\pi} &= 0 \\ \underbrace{= 0, \delta X|_{\tau_1, \tau_2} = 0} &\quad \hookrightarrow \text{Neumann bc: } X^{\mu\prime}|_{\sigma=0, \pi} = 0 \end{aligned}$$

Verifying that X^μ satisfy eom:

$$-\ddot{X}^0 + X^{0\prime\prime} = -(0) + (0) = 0 \quad \checkmark$$

$$-\ddot{X}^1 + X^{1\prime\prime} = -(-B \cos \tau \cos \sigma) + (-B \cos \tau \cos \sigma) = 0 \quad \checkmark$$

$$-\ddot{X}^2 + X^{2\prime\prime} = -(-B \sin \tau \cos \sigma) + (-B \sin \tau \cos \sigma) = 0 \quad \checkmark$$

$$-\ddot{X}^3 + X^{3\prime\prime} = 0 \quad \checkmark$$



Verifying Neumann BC:

$$X^{0\prime}|_{\sigma=0, \pi} = 0 \quad \checkmark$$

$$X^{1\prime}|_{\sigma=0, \pi} = -B \cos \tau \sin \sigma|_{\sigma=0, \pi} = 0 \quad \checkmark$$

$$X^{2\prime}|_{\sigma=0, \pi} = -B \sin \tau \sin \sigma|_{\sigma=0, \pi} = 0 \quad \checkmark$$

$$X^{3\prime}|_{\sigma=0, \pi} = 0 \quad \checkmark$$



Speed of end of string

$$\begin{aligned} v &= \frac{dX^i(\text{ends})}{dX^0} \quad i>0 \\ &= \frac{dX^i/d\tau}{dX^0/d\tau} = \begin{cases} i=1 & -B \sin \tau \cos \sigma / B \\ i=2 & -B \cos \tau \cos \sigma / B \\ i>2 & 0 \end{cases} \bigg|_{\sigma=0, \pi} = \begin{cases} i=1 & \mp \sin \tau \\ i=2 & \mp \cos \tau \\ i>2 & 0 \end{cases} \end{aligned}$$

$$|v| = \sqrt{\sin^2 \tau + \cos^2 \tau} = 1 = c$$



(ii) Compute the energy $E = P^0$ and angular momentum J of the string. Use the result to show that $\frac{E}{|J|} = 2\pi T = \frac{1}{\alpha'}$.

$$P_\alpha^\mu = T \partial_\alpha X^\mu \rightarrow P^0 = \int ds T \dot{X}^0 = T \int_0^{2\pi} ds B = \pi B T$$

$$J_\alpha^\mu = T (X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu) \rightarrow J^z = T \int ds (X^1 \dot{X}^2 - X^2 \dot{X}^1) \quad \text{all other } J^i \text{ are 0.}$$

$$= T \int ds (-B^2 \cos^2 \tau \cos^2 \sigma + B^2 \sin^2 \tau \cos^2 \sigma)$$

$$= T B^2 \int ds \cos^2 \tau \cos^2 \sigma = \frac{\pi T B^2}{2}$$

$$\frac{E}{|J|} = \frac{\pi^2 B^2 T^2}{\pi B^2 T/2} = 2\pi T = 2\pi \left(\frac{1}{2\pi\alpha'} \right) = \frac{1}{\alpha'}$$

(iii) Show that the constraint $T_{\alpha\beta} = 0$ can be written as $(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0$, $\partial_\tau X \cdot \partial_\sigma X = 0$ and that this constraint is satisfied by the above solution.

$$T_{\alpha\beta} = -\frac{1}{2} \frac{1}{T} \frac{\delta S}{\delta g^{\alpha\beta}} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} \eta^{\mu\nu} \partial_\mu X \cdot \partial_\nu X = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} (-\dot{X}^2 + X'^2) = 0$$

For $\mu=\nu=0$, $(\partial_\tau X)^2 + \frac{1}{2} (-(\partial_\tau X)^2 + (\partial_\sigma X)^2) = \frac{1}{2} ((\partial_\tau X)^2 + (\partial_\sigma X)^2) = 0$

$\alpha=\beta=1$, $(\partial_\sigma X)^2 - \frac{1}{2} (-(\partial_\tau X)^2 + (\partial_\sigma X)^2) = \frac{1}{2} ((\partial_\tau X)^2 + (\partial_\sigma X)^2) = 0$

$\alpha=0, \beta=1$, $\partial_\tau X \cdot \partial_\sigma X = 0$

$$\dot{X} \cdot \dot{X} + X' \cdot X' = (-B^2 + B^2 \sin^2 \tau \cos^2 \sigma + B^2 \cos^2 \tau \cos^2 \sigma) + (-0 + B^2 \cos^2 \tau \sin^2 \sigma + B^2 \sin^2 \tau \sin^2 \sigma)$$

$$= B^2 \left(-1 + \sin^2 \tau (\cos^2 \sigma + \sin^2 \sigma) + \cos^2 \tau (\sin^2 \sigma) \right) = 0$$

$$\dot{X} \cdot X' = -B \cdot 0 + (-B \sin \tau \cos \sigma) (-B \cos \tau \sin \sigma) + (-B \cos \tau \cos \sigma) (B \sin \tau \sin \sigma)$$

$$= 2B^2 \sin \tau \cos \tau \sin \sigma \cos \sigma \Big|_{\sigma=0, \pi} = 0$$

2.2 Consider the open string w/ trajectory $X^0 = 3\tau c$, $X^1 = A \cos 3\tau c \cos 3\sigma$, $X^2 = A \sin 3\tau c \cos 3\sigma$ and assume $\eta_{\alpha\beta} = \eta_{\mu\nu}$.

(i) Determine a, b st. $T_{\alpha\beta} = 0$. Express the solution as $X^\mu = X_L^\mu(\sigma^-) + X_R^\mu(\sigma^+)$. Determine the bc. satisfied.

$$\dot{X}^2 + X'^2 = (-9A^2 + 9A^2 \sin^2(3\tau c) \cos^2(3\sigma) + 9A^2 \cos^2(3\tau c) \cos^2(3\sigma)) + (9A^2 \cos^2(3\tau c) \sin^2(3\sigma) + 9A^2 \sin^2(3\tau c) \sin^2(3\sigma)) \Rightarrow a^2 = b^2 = 9 \Rightarrow a = b = \pm 3$$

Let $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$, then

$$X^0 = 3A\tau c = \frac{3}{2}A(\sigma^+ + \sigma^-)$$

$$\rightarrow X^0|_{\sigma=\pi} \neq \text{const.}$$

$$\partial_\sigma X^0|_{\sigma=\pi} = 0 \Rightarrow \text{Neumann}$$

$$X^1 = A \cos(3\tau c) \cos(3\sigma) = \frac{1}{2}A(\cos(3\sigma^+) + \cos(3\sigma^-))$$

$$\rightarrow X^1|_{\sigma=\pi} = \text{const.}$$

$$\partial_\sigma X^1|_{\sigma=\pi} = 3A \cos(3\tau c) \sin(3\sigma)|_{\sigma=\pi} = 0 \Rightarrow D.N.$$

$$X^2 = A \sin(3\tau c) \cos(3\sigma) = \frac{1}{2}A(\sin(3\sigma^+) + \sin(3\sigma^-))$$

$$\rightarrow X^2|_{\sigma=\pi} \neq \text{const.}$$

$$\partial_\sigma X^2|_{\sigma=\pi} = 2A \sin(3\tau c) \sin(3\sigma)|_{\sigma=\pi} = 0 \Rightarrow D.N.$$

(iii) Compute the con. momentum and angular momentum and show that they are conserved. What do you obtain for the relation of E and J? Comment on your result.

$$P = \int ds P_0 = T \int ds \dot{X} = 3\pi R T //$$

$$P^i = \int ds P_0^i = T \int ds \partial_\tau X^i = -T \int ds \dot{X}^i \sin(3\sigma) \cos(3\sigma) = -\frac{3\pi R T}{3} \sin(3\sigma) \sin(3\sigma) \Big|_{0, 2\pi} = 0 //$$

$$P^z = 0 //$$

$$J^z = T \int ds X^i \dot{X}^i - X^i \dot{X}^i = T \int ds 3R^2 \cos^2(3\sigma) \dot{\sigma}^2 + 3R^2 \sin^2(3\sigma) \dot{\sigma}^2 = 3\pi R T \left(\underbrace{\sin^2(3\sigma) + \cos^2(3\sigma)}_1 \right) \int ds \underbrace{\dot{\sigma}^2(3\sigma)}_{\frac{1}{2} \cos(6\sigma) + \frac{1}{2}} = \frac{3}{2} \pi R T //$$

$$\frac{E}{J} = \frac{9 \pi R^2 T^2}{3 \pi R T^2} = 6\pi T = 6\pi \left(\frac{1}{2\pi\alpha'} \right) = \frac{3}{\alpha'} = \frac{3}{2\ell_s^2} \Rightarrow \ell_s^2 = 6 \frac{3}{E^2} \left(\text{before } \ell_s^2 = 2 \frac{3}{E^2} \right)$$



2.3 Compute the mode expansion of an open string w/ Neumann b.c. for the coordinates $X^0, \dots, X^d$, while X^{25} satisfies:

(i) Dirichlet b.c. at both ends $X^{25}(0, \tau) = X_0^{25}$ and $X^{25}(\pi, \tau) = X_\pi^{25}$

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \quad \mu = 0, 1, \dots, 24$$

General solution to wave equation is $X^{25}(\tau, \sigma) = \psi(\tau, \sigma) = \psi_L(\tau, \sigma) + \psi_R(\tau, \sigma)$

B.c. $\psi_L(\tau) + \psi_R(\tau) = \psi_0$ and $\psi_L(\tau, \pi) + \psi_R(\tau, \pi) = \psi_\pi$

$$\Rightarrow \psi_L'(\tau) = -\psi_R'(\tau) \quad \text{and} \quad \psi_L'(\tau, \pi) + \psi_R'(\tau, \pi) = \psi_L'(\tau, \pi) - \psi_L'(\tau, \pi) = 0$$

$$\psi_L'(\tau + 2\pi) = \psi_L'(\tau)$$

$$\psi_L(u) = \ell_s \sum_n \alpha_n e^{-in u} \quad \text{w/ } (\alpha_n)^* = \alpha_{-n} \quad (\text{Reality condition})$$

$\Rightarrow$

$$\psi_L(u) = \int du \ell_s \sum_n \alpha_n e^{-in u} = \int du \ell_s \alpha_0 + \int du \ell_s \sum_{n \neq 0} \alpha_n e^{-in u} = \ell_s \alpha_0 u + \ell_s \sum_{n \neq 0} \frac{1}{-in} \alpha_n e^{-in u} + \text{const.} = c_1 + \ell_s \alpha_0 u + i\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in u}$$

$$\psi_R(u) = \int du -\psi_L'(u) = c_2 - \ell_s \alpha_0 u - i\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in u}$$

$\tau +$

$$\psi(\tau, \sigma) = \psi_L(\tau, \sigma) + \psi_R(\tau, \sigma) = c_1 + c_2 + 2\ell_s \alpha_0 \sigma + i\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n \left(\underbrace{e^{-in(\tau+\sigma)} - e^{-in(\tau-\sigma)}}_{e^{-in\tau} \left(\frac{e^{-in\sigma} - e^{in\sigma}}{2i} \right)} \right) = c + 2\ell_s \alpha_0 \sigma + 2\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in\tau} \sin(n\sigma) \quad (2.5)$$

$$\psi(\tau, \sigma) = c = \psi_0$$

$$\psi(\tau, \pi) = \psi_0 + 2\pi \ell_s \alpha_0 = \psi_\pi \Rightarrow \alpha_0 = \frac{\psi_\pi - \psi_0}{2\pi \ell_s}$$

$$\Rightarrow X^{25}(\tau, \sigma) = X_0^{25} + \frac{1}{\pi} (X_\pi^{25} - X_0^{25}) \sigma + 2\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in\tau} \sin(n\sigma) //$$

$$\begin{aligned}
 P^{05} &= T \int ds \partial_\tau X^{05} = T \int ds \left(-2i\alpha' \sum_{n \neq 0} e^{-in\tau} \sin(n\sigma) \right) = -2i\alpha' T \sum_{n \neq 0} e^{in\tau} \left[\frac{-1}{n} \cos(n\sigma) \right]_0^\pi = 2i\alpha' T \left(\sum_{n \neq 0} (-1)^n \frac{1}{n} e^{-in\tau} - \sum_{n \neq 0} \frac{1}{n} e^{in\tau} \right) \\
 &= -4i\alpha' T \sum_{\substack{n \neq 0 \\ n \text{ odd}}} \frac{1}{n} e^{-in\tau} = -4i\alpha' T \sum_{n=1,3,5,\dots} \frac{1}{2n+1} e^{i(2n+1)\tau}
 \end{aligned}$$


(ii) Dirichlet on one end $X^{25}(0, \tau) = X_0^{25}$ and Neumann on other end $\partial_\sigma X^{25}(\pi, \tau) = 0$.

Similar to the above:

$$Y(\tau, \sigma) = Y_L(\tau, \sigma) + Y_R(\tau, \sigma)$$

$$\begin{aligned}
 Y(\sigma, 0) &= \underbrace{Y_L(\tau) + Y_R(\tau)}_{Y_0} \quad \text{and} \quad \partial_\sigma Y(\tau, \sigma) \Big|_{\pi} = \underbrace{Y_L'(\tau) - Y_R'(\tau)}_{\pi} = 0 \\
 Y_L'(\sigma) &= -Y_R'(\sigma) \qquad Y_L'(\sigma+2\pi) = Y_R'(\sigma) = -Y_L'(\sigma) \Rightarrow Y_L'(\sigma+\pi) = Y_L'(\sigma)
 \end{aligned}$$

$$Y_L(\sigma) = \alpha_0 \sum_n \alpha_n e^{-i\frac{\sigma}{2} n} = \alpha_0 \sum_n \alpha_n e^{-i\frac{1}{2} n \sigma} = \alpha_0 \sum_n \alpha_{2n} e^{-i(2n)\sigma/2}$$

$$\begin{aligned}
 \alpha_n &= \frac{1}{\pi} \int_0^\pi d\sigma Y_L'(\sigma) e^{i\frac{n}{2} \sigma} \\
 \alpha_{2n} &= \frac{1}{4\pi} \int_0^{2\pi} d\sigma Y_L'(\sigma) e^{in\sigma} = \frac{1}{4\pi} \left\{ \int_0^\pi d\sigma Y_L'(\sigma) e^{in\sigma} + \int_\pi^{2\pi} d\sigma Y_L'(\sigma) e^{in\sigma} \right\} = \frac{1}{4\pi} (0) \\
 &\quad = \frac{1}{4\pi} \left\{ \int_0^\pi d\sigma Y_L'(\sigma) e^{in\sigma} - \int_0^\pi d\sigma Y_L'(\sigma-2\pi) e^{in\sigma} \right\} = \frac{1}{4\pi} \left(\int_0^\pi d\sigma Y_L'(\sigma) (1 - e^{-2in\sigma}) \right)
 \end{aligned}$$

$$\begin{aligned}
 Y_L(\sigma) &= c_1 + \alpha_0 \int d\sigma \sum_n \alpha_{2n} e^{-i(2n)\sigma/2} \\
 &= c_1 + 2i\alpha' \sum_n \frac{1}{(2n+1)} \alpha_{2n+1} e^{-i(2n+1)\sigma/2}
 \end{aligned}$$

$$\begin{aligned}
 Y_R(\sigma) &= c_2 - \dots \\
 \Rightarrow Y(\tau, \sigma) &= c + 2i\alpha' \sum_n \frac{1}{(2n+1)} \alpha_{2n+1} e^{-i(2n+1)\tau/2} \left(e^{i(2n+1)\sigma/2} - e^{i(2n+1)\pi\sigma/2} \right) \\
 &= Y_0 + 4\alpha' \sum_n \frac{1}{(2n+1)} \alpha_{2n+1} e^{-i(2n+1)\tau/2} \sin\left(\frac{(2n+1)\sigma}{2}\right)
 \end{aligned}$$


2.4 Consider the bosonic string in the light-cone gauge.

(i) Find the mass squared of the following on-shell open-string states: $|p_1\rangle = \alpha_{-1}^\dagger |0; k\rangle$ $|p_2\rangle = \alpha_{-1}^\dagger \alpha_{-1}^\dagger |0; k\rangle$
 $|p_3\rangle = \alpha_{-2}^\dagger |0; k\rangle$ $|p_4\rangle = \alpha_{-1}^\dagger \alpha_{-2}^\dagger |0; k\rangle$

Light-cone coordinates: $X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^1) \Rightarrow u \cdot v = -u^+ v^- - u^- v^+ + \sum_i u^i v^i \left\{ \begin{array}{l} \sigma^+ = \tau + \sigma, d\sigma^+ = d\tau + d\sigma, d\tau = \frac{1}{2}(d\sigma^+ - d\sigma^-) \\ \sigma^- = \tau - \sigma, d\sigma^- = d\tau - d\sigma, d\tau = \frac{1}{2}(d\sigma^+ + d\sigma^-) \end{array} \right.$

Energy-momentum tensor is zero

$$\begin{aligned}
 T_{\tau\tau} &= -\frac{2}{\alpha'} \frac{1}{4\pi} \frac{1}{2} \eta_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu = \partial_\tau X \cdot \partial_\tau X - \frac{1}{2} \eta_{\mu\nu} \partial_\tau X^\mu \partial_\sigma X^\nu \Big|_{\eta_{\mu\nu}} = \partial_\tau X \cdot \partial_\tau X - \frac{1}{2} \eta_{\mu\nu} (-\dot{X}^\mu + X'^\mu) = 0 \quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\tau^+ + \sigma^+ = -\frac{1}{4}(\sigma^+ + \sigma^-)^2 \\
 &\quad = -\sigma^+ \sigma^- = \left(\eta_{+-} \eta_{-+} \right) + \frac{1}{4}(\sigma^- - \sigma^+)^2 \\
 T_{\sigma\sigma} &= \partial_\sigma X \cdot \partial_\sigma X - \frac{1}{2} \eta_{\mu\nu} (-\partial_\sigma X^\mu \partial_\tau X^\nu) = \partial_\sigma X \cdot \partial_\sigma X = \frac{1}{4} (\partial_\tau X + \partial_\sigma X)^2 = \frac{1}{4} (\dot{X}^2 + X'^2) + \frac{1}{2} \dot{X} \cdot X' = 0 \quad = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 T_{\tau\sigma} &= \partial_\tau X \cdot \partial_\sigma X = 0 \quad \frac{1}{2} T_{\sigma\sigma} = 0 \quad \frac{1}{2} T_{\tau\sigma} = 0
 \end{aligned}$$

$$\sqrt{T_-} = \frac{1}{2}(\partial_\tau - \partial_\sigma)(x^\mu + \ell_s^2 p^\mu \tau + i \ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)) = \frac{1}{2} \ell_s^2 p^\mu + \frac{i}{2} \ell_s \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) + \frac{1}{2} i \ell_s \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \sin(n\sigma)$$

$$= \frac{1}{2} \ell_s^2 p^\mu + \frac{1}{2} \ell_s \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} (\cos(n\sigma) + i \sin(n\sigma)) = \frac{1}{2} \ell_s \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$T_- = \frac{1}{4} \ell_s^2 \left(\sum_n \alpha_n^\mu e^{-in(\tau-\sigma)} \right) \cdot \left(\sum_n \alpha_n^\mu e^{-in(\tau-\sigma)} \right) = \frac{1}{4} \ell_s^2 \sum_n \sum_m \alpha_{n-m}^\mu \alpha_n^\mu e^{-in(\tau-\sigma) - im(\tau-\sigma)} = \frac{1}{4} \ell_s^2 \sum_n \sum_m \alpha_{n-m}^\mu \alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$= \frac{1}{4} \ell_s^2 \sum_n \left(\sum_m \alpha_{n-m}^\mu \alpha_m^\mu \right) e^{-in(\tau-\sigma)} = 0 \quad \Rightarrow \quad L_m = 0 \quad \forall m$$

Lm Virasoro generators

$$L_0 = \frac{1}{2} \sum_{n \neq 0} \alpha_n^\mu \alpha_{-n}^\mu = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_n^\mu = \frac{1}{2} \ell_s^2 p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_n^\mu = 0 \quad \Rightarrow \quad M^2 = \frac{2}{\ell_s^2} \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_n^\mu = \frac{2}{\ell_s^2} \sum_{n=1}^{\infty} \left(-\alpha_{-n}^\mu \alpha_n^\mu - \alpha_{-n}^\mu \alpha_n^\mu + \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i \right)$$

$$= \frac{2}{\ell_s^2} \left(\sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - \underbrace{\sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_n^\mu + \alpha_{-n}^\mu \alpha_n^\mu)}_a \right) = \frac{2}{\ell_s^2} (N - a)$$

Thus, $[\alpha_{-a}, \alpha_{a+2}] \Rightarrow$ needs to be massless!

$$M^2 |q_1\rangle = \frac{2}{\ell_s^2} (N-a) \alpha_{-1}^\mu \alpha_{1+2}^\mu |q_1\rangle = \frac{2}{\ell_s^2} (1-a) = 0 \Rightarrow a=1$$

$$N |q_1\rangle = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i \alpha_{1+2}^\mu |q_1\rangle = \alpha_{-1}^\mu \alpha_{1+2}^\mu |q_1\rangle = \alpha_{-1}^\mu \alpha_3^\mu |q_1\rangle$$

$$= \alpha_{-1}^\mu \alpha_3^\mu + (n) \eta^{\mu\nu} \delta_{n+1,0}$$

$$M^2 |q_2\rangle = \frac{2}{\ell_s^2} (N-1) \alpha_{-1}^\mu \alpha_{1+2}^\mu |q_2\rangle = \frac{2}{\ell_s^2} |q_2\rangle \quad \leftarrow M^2 [\alpha_{-a}]_{D=26} - 1 \text{ det in } D=26$$

massive graviton

$$N |q_2\rangle = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i \alpha_{1+2}^\mu |q_2\rangle = \sum_{i=1}^{D-2} \sum_n \alpha_{-n}^i \alpha_{-1}^\mu \alpha_n^i \alpha_2^\mu |q_2\rangle + \alpha_{-1}^\mu \alpha_3^\mu |q_2\rangle$$

$$\alpha_{-1}^\mu \alpha_3^\mu + n \eta^{\mu\nu} \delta_{n+1,0} \quad \alpha_{-1}^\mu \alpha_3^\mu + n \eta^{\mu\nu} \delta_{n+1,0}$$

$$M^2 |q_3\rangle = \frac{1}{\ell_s^2} (N-1) |q_3\rangle = \frac{1}{\ell_s^2} |q_3\rangle$$

$$N |q_3\rangle = \sum_i \sum_n \alpha_{-n}^i \alpha_n^i \alpha_{1+2}^\mu |q_3\rangle = 3 \alpha_{-1}^\mu \alpha_3^\mu |q_3\rangle = 3 |q_3\rangle$$

$$\alpha_{-1}^\mu \alpha_3^\mu + n \eta^{\mu\nu} \delta_{n+1,0}$$

$$M^2 |q_4\rangle = \frac{6}{\ell_s^2} |q_4\rangle$$

(ii) Find M^2 of the following on-shell closed-string states: $|q_1\rangle = \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\mu |0; k\rangle$ $|q_2\rangle = \alpha_{-1}^\mu \alpha_{-1}^\mu \tilde{\alpha}_{-2}^\mu |0; k\rangle$.

$$M^2 = \frac{4}{\ell_s^2} \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_n^\mu + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\mu) = \frac{4}{\ell_s^2} \left(- \underbrace{\sum_n (\alpha_{-n}^\mu \alpha_n^\mu + \alpha_{-n}^\mu \alpha_n^\mu)}_a - \underbrace{\sum_n (\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\mu + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\mu)}_a + \underbrace{\sum_n \sum_i \alpha_{-n}^i \alpha_n^i}_N + \underbrace{\sum_n \sum_i \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i}_{\tilde{N} = N \text{ level matching condition}} \right) = \frac{8}{\ell_s^2} (N-a) = \frac{8}{\ell_s^2} (\tilde{N}-a)$$

$$M^2 |q_1\rangle = \frac{8}{\ell_s^2} (N-a) \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\mu |0; k\rangle = 0$$

$$M^2 |q_2\rangle = \frac{8}{\ell_s^2} (N-1) \alpha_{-1}^\mu \alpha_{-1}^\mu \tilde{\alpha}_{-2}^\mu |0; k\rangle = \frac{8}{\ell_s^2} |q_2\rangle$$

(iii) What can you say about the following closed-string state $\alpha_{-1}^\mu \tilde{\alpha}_{-2}^\mu |0; k\rangle$?

Violates level-matching $\rightarrow T_{00} \neq 0$

2.5 For an open string w/ Neumann boundary condition and commutation relations for the modes to show

$$[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = [P^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = 0 \quad \& \quad [X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = i\eta^{\mu\nu} \delta(\sigma - \sigma')$$

$$\begin{aligned} X^\mu(\tau, \sigma) &= x^\mu + \frac{1}{2\alpha'} p^\mu \tau + i\alpha' \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \quad \left[\alpha_n, \alpha_m \right] = 0, \eta^{\mu\nu} \delta_{n+m, 0} = 0 \\ \Rightarrow [X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] &= \left[\left(x^\mu + \frac{1}{2\alpha'} p^\mu \tau + i\alpha' \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \right), \left(x^\nu + \frac{1}{2\alpha'} p^\nu \tau' + i\alpha' \sum_{m \neq 0} \frac{1}{m} \alpha_m^\nu e^{-im\tau'} \cos(m\sigma') \right) \right] \\ &= -\alpha'^2 \sum_{n, m \neq 0} \frac{1}{nm} [\alpha_n^\mu, \alpha_m^\nu] e^{-i(n\tau + m\tau')} \cos(n\sigma) \cos(m\sigma') = +\alpha'^2 \sum_{n \neq 0} \frac{1}{n^2} \underbrace{\cos(n\sigma) \cos(n\sigma')}_{\text{odd}} = 0 \end{aligned}$$

$$\begin{aligned} P^\mu(\tau, \sigma) &= T \dot{X}^\mu = T x'^\mu + T \dot{\alpha}' \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \\ \Rightarrow \text{Similarly for } [P^\mu, P^\nu] &\sim \alpha'^2 \sum_{n \neq 0} n \underbrace{\cos(n\sigma) \cos(n\sigma')}_{\text{odd}} = 0 \end{aligned}$$

$$[X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = iT \sum_{n, m \neq 0} \frac{1}{nm} [\alpha_n^\mu, \alpha_m^\nu] e^{-i(n\tau + m\tau')} \cos(n\sigma) \cos(m\sigma') = i \left(\frac{1}{2\pi\alpha'} \right) \alpha'^2 \sum_{n \neq 0} \cos(n\sigma) \cos(n\sigma') \eta^{\mu\nu} = i\eta^{\mu\nu} \delta(\sigma - \sigma')$$

2.6 For an open string, use the canonical commutation relations to verify the Poincaré algebra.

$$[p^\mu, p^\nu] = \int d\sigma d\sigma' \underbrace{[P^\mu(\sigma, \tau), P^\nu(\sigma', \tau)]}_0 = 0$$

$$\begin{aligned} [p^\mu, J^{\mu\nu}] &= [p^\mu, x^\mu p^\nu - x^\nu p^\mu - i \sum_{n \neq 0} \frac{1}{n} (\alpha_n^\mu \alpha_n^\nu - \alpha_n^\nu \alpha_n^\mu)] = [p^\mu, x^\mu] p^\nu - [p^\mu, x^\nu] p^\mu - i \sum_{n \neq 0} \frac{1}{n} ([p^\mu, \alpha_n^\mu \alpha_n^\nu] - [p^\mu, \alpha_n^\nu \alpha_n^\mu]) \\ &= \frac{1}{\pi} \int d\sigma d\sigma' \underbrace{[P^\mu(\tau, \sigma), X^\mu(\tau, \sigma')] p^\nu}_{-i\eta^{\mu\nu} \delta(\sigma - \sigma')} - \underbrace{[P^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] p^\mu}_{-i\eta^{\mu\nu} \delta(\sigma - \sigma')} p^\mu \\ &= \frac{1}{\pi} \int d\sigma \underbrace{i\eta^{\mu\nu} p^\sigma}_{\text{since } \sum \alpha_n \text{ are internal oscillations, this should vanish...}} - i\eta^{\mu\nu} p^\sigma = i\eta^{\mu\nu} p^\sigma - i\eta^{\mu\nu} p^\sigma = 0 \end{aligned}$$

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= [x^\mu p^\nu - x^\nu p^\mu, x^\rho p^\sigma - x^\sigma p^\rho] - \sum_{n, m \neq 0} \frac{1}{nm} [\alpha_n^\mu \alpha_m^\nu - \alpha_n^\nu \alpha_m^\mu, \alpha_n^\rho \alpha_m^\sigma - \alpha_n^\sigma \alpha_m^\rho] \\ &= [\alpha_n^\mu, \alpha_m^\rho] \alpha_n^\nu \alpha_m^\sigma - \alpha_n^\nu [\alpha_n^\mu, \alpha_m^\rho] \alpha_n^\sigma + \alpha_n^\nu \alpha_m^\rho [\alpha_n^\mu, \alpha_m^\sigma] - \alpha_n^\sigma [\alpha_n^\mu, \alpha_m^\rho] \alpha_n^\nu \\ &= \eta^{\mu\rho} \left(-\frac{m \delta_{-n-m, 0}}{n=-m} \alpha_n^\nu \alpha_m^\sigma + \frac{m \delta_{-m-n, 0}}{n=m} \alpha_n^\nu \alpha_m^\sigma - \frac{m \delta_{-m-n} \alpha_n^\mu \alpha_m^\rho}{n=-m} + \frac{m \delta_{-n-m, 0}}{n=-m} \alpha_n^\mu \alpha_m^\rho \right) \\ &= -m \alpha_n^\mu \alpha_{-n-m}^\nu + m \alpha_n^\nu \alpha_{-n-m}^\mu - m \alpha_n^\mu \alpha_{-m-n}^\rho + m \alpha_n^\rho \alpha_{-m-n}^\mu = 0 \quad \text{Similarly for all other terms...} \end{aligned}$$

$$\begin{aligned} ([AB, CD] + A[B, C]D + A[B, C]D + C[A, D]B + [A, C]DB) \\ = [x^\mu p^\nu, x^\rho p^\sigma] - [x^\mu p^\nu, x^\sigma p^\rho] - [x^\nu p^\mu, x^\rho p^\sigma] + [x^\nu p^\mu, x^\sigma p^\rho] \quad [x^\mu, p^\nu] = \frac{1}{\pi} \int d\sigma d\sigma' [X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = \frac{1}{\pi} \int d\sigma d\sigma' i\eta^{\mu\nu} \delta(\sigma - \sigma') = i\eta^{\mu\nu} \\ = x^\mu [p^\nu, x^\rho] p^\sigma + x^\mu [x^\rho, p^\sigma] p^\nu - x^\nu [p^\mu, x^\rho] p^\sigma - x^\nu [x^\rho, p^\sigma] p^\mu + x^\nu [p^\mu, x^\sigma] p^\rho + x^\nu [x^\sigma, p^\rho] p^\mu \\ = i\eta^{\nu\rho} \underbrace{(-x^\mu p^\sigma + x^\sigma p^\mu)}_{-J^{\rho\sigma}} + i\eta^{\mu\rho} \underbrace{(x^\nu p^\sigma - x^\sigma p^\nu)}_{-J^{\nu\sigma}} + i\eta^{\mu\sigma} \underbrace{(x^\nu p^\rho - x^\rho p^\nu)}_{J^{\nu\rho}} + i\eta^{\nu\sigma} \underbrace{(-x^\mu p^\rho + x^\rho p^\mu)}_{J^{\mu\rho}} \end{aligned}$$

2.7 Derive $J^{\mu\nu}$ for a closed string.

$$X^\mu(\tau, \sigma) = x^\mu + \ell_s^2 p^\mu \tau + \frac{i\ell_s}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right)$$

$$\dot{X}^\mu(\tau, \sigma) = \ell_s^2 p^\mu + \ell_s \sum_{n \neq 0} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right)$$

$$\begin{aligned} X^\mu \dot{X}^\nu &= \left(x^\mu + \ell_s^2 p^\mu \tau + \frac{i\ell_s}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right) \right) \left(\ell_s^2 p^\nu + \ell_s \sum_{n \neq 0} \left(\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)} \right) e^{-2in\tau} \right) \\ &= \ell_s^2 x^\mu p^\nu + \ell_s^2 \tau p^\mu p^\nu + \ell_s (x^\mu + \ell_s^2 p^\mu \tau) \sum_{n \neq 0} \left(\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)} \right) e^{-2in\tau} + \frac{i\ell_s^2}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right) e^{-2in\tau} \\ &\quad + \frac{i\ell_s^2}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right) \left(\alpha_n^\nu e^{-2in\tau} + \tilde{\alpha}_n^\nu e^{-2in\tau} \right) e^{-2in\tau} \end{aligned}$$

Consider $\int_0^\pi d\sigma e^{2im\tau} = 0 \quad m \in \mathbb{Z}, m \neq 0$

$$= \ell_s^2 x^\mu p^\nu + \frac{i\ell_s^2}{2} \sum_{n, m \neq 0} \frac{1}{n} \left(\alpha_n^\mu \alpha_m^\nu e^{\frac{2i(n-m)\tau}{\ell_s^2}} + \alpha_n^\mu \tilde{\alpha}_m^\nu e^{\frac{2i(n+m)\tau}{\ell_s^2}} + \tilde{\alpha}_n^\mu \alpha_m^\nu e^{\frac{2i(-n-m)\tau}{\ell_s^2}} + \tilde{\alpha}_n^\mu \tilde{\alpha}_m^\nu e^{\frac{2i(-n+m)\tau}{\ell_s^2}} \right) e^{-2i(n+m)\tau}$$

$$= \ell_s^2 x^\mu p^\nu + \frac{i\ell_s^2}{2} \sum_{m \neq 0} \frac{1}{m} \left(\alpha_m^\mu \alpha_{-m}^\nu + \alpha_m^\mu \tilde{\alpha}_{-m}^\nu e^{-4im\tau} + \tilde{\alpha}_m^\mu \alpha_{-m}^\nu e^{-4im\tau} + \tilde{\alpha}_m^\mu \tilde{\alpha}_{-m}^\nu \right)$$

$$T_0 = \frac{1}{2\pi\alpha'} = \frac{1}{\pi\ell_s^2} \quad \alpha' = \frac{1}{2}\ell_s^2$$

$$\begin{aligned} J^\mu &= T \int_0^\pi d\sigma \left(\dot{X}^\mu \dot{X}^\nu - X^\mu \ddot{X}^\nu \right) = T \int_0^\pi d\sigma \left(\ell_s^2 x^\mu p^\nu - \ell_s^2 x^\mu p^\nu + \frac{i\ell_s^2}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\mu \alpha_{-n}^\nu + \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu \right) \right) \\ &= x^\mu p^\nu - x^\mu p^\nu + \frac{1}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\mu \alpha_{-n}^\nu + \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu \right) \end{aligned}$$

2.8 The expression for J^μ in Ex. 2.10, is appropriate for covariant quantisation. What are the formulas for the case of light-cone gauge quantisation?

Review of light-cone gauge:

In $\eta_{\mu\nu} = \eta_{\mu\rho} \eta_{\rho\nu}$ gauge the con are: $\partial_\alpha \partial^\alpha X^\mu = (-\partial_-^2 + \partial_+^2) X^\mu = 0$

$$T_{00} = T_{11} = \frac{1}{2} (\dot{X}^2 + X'^2) = 0 \quad \text{and} \quad T_{01} = T_{10} = \dot{X} \cdot X' = 0 \quad \text{Virasoro constraints.}$$

Want to solve Virasoro constraints explicitly and then only quantise free dof.

Consider the metric in light-cone coordinates $\sigma^2 = \tau^2 - \sigma^2$

$$ds^2 = -d\tau^2 + d\sigma^2 = -2d\sigma^+ d\sigma^-$$

$$\Rightarrow \text{for } \tilde{\sigma}^+ = f(\sigma^+), \tilde{\sigma}^- = g(\sigma^+): \quad d\sigma^2 \rightarrow -2d\tilde{\sigma}^+ d\tilde{\sigma}^- = -2f'(\sigma^+) g'(\sigma^+) d\sigma^+ d\sigma^-$$

can be removed via Weyl transf. $\rightarrow$ additional gauge freedom!

So we are free to define some $\tilde{\tau}$ as

$$\tilde{\tau} = \frac{1}{2} (\tilde{\sigma}^+ + \tilde{\sigma}^-) = \frac{1}{2} (f(\tau+\sigma) + g(\tau-\sigma)) \rightarrow \text{free wave eq. just like } X^\mu!$$

The light-cone gauge is the following choice for $\tilde{\tau}$:

$$X^+ = x^+ + \ell_s^2 p^+ \tilde{\tau} \quad \Leftrightarrow \quad \alpha_n^+ = 0 \quad \forall n \neq 0.$$

Now considering the Virasoro constraints again.

$$\dot{X}^\mu \cdot \dot{X}^\mu = 0 \Rightarrow -2\dot{X}^\mu \dot{X}^\mu + \sum_{n=1}^{\infty} \dot{X}^\mu \dot{X}^\mu - 2\dot{X}^\mu \dot{X}^\mu + \sum_{n=1}^{\infty} \dot{X}^\mu \dot{X}^\mu = -2\ell_s^2 p^\mu \dot{X}^\mu + \dot{X}_i \dot{X}_i + \dot{X}_i \dot{X}_i = 0 \Rightarrow \dot{X}^\mu = \frac{1}{2\ell_s^2 p^\mu} (\dot{X}_i \dot{X}_i + \dot{X}_i \dot{X}_i)$$

$$\dot{X}^\mu \cdot X^\mu = 0 \Rightarrow -\dot{X}^\mu X^\mu - \dot{X}^\mu X^\mu + \dot{X}_i X_i + \dot{X}_i X_i = -\ell_s^2 p^\mu X^\mu + \dot{X}_i X_i = 0 \Rightarrow X^\mu = \frac{1}{\ell_s^2 p^\mu} (\dot{X}_i X_i)$$

Therefore X^μ is completely determined and not indep.: $X^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau + \frac{\partial X^\mu}{\partial \sigma} d\sigma$

$$J^\mu = x^\mu p^\mu - \dot{x}^\mu p^\mu + i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n^\mu \alpha_{-n}^\mu - \alpha_{-n}^\mu \alpha_n^\mu)$$

$$\Rightarrow J^\mu = x^\mu p^\mu - \dot{x}^\mu p^\mu, \quad J^i = x^i p^i - \dot{x}^i p^i$$

$$\begin{aligned} J^i &= x^i p^i - \dot{x}^i p^i + i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n^i \alpha_{-n}^i - \alpha_{-n}^i \alpha_n^i) = x^i p^i - \dot{x}^i p^i \\ &= i \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{1}{p^0} \frac{1}{2} \sum_{j=1}^{\infty} \alpha_n^j \alpha_{-n-j}^i - \frac{1}{2p^0} \sum_{j=1}^{\infty} \alpha_{-n-j}^i \alpha_n^j \right\} \\ &= i \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{2p^0} \left((n-2n) \alpha_{-n}^i \alpha_n^i + n \alpha_{-n}^i \alpha_n^i \right) = 0 \end{aligned}$$

$$J^j = x^j p^j - \dot{x}^j p^j + i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n^j \alpha_{-n}^j - \alpha_{-n}^j \alpha_n^j)$$

2.9) Show that the Lorentz generators commute w/ all Virasoro generators: $[L_m, J^\mu] = 0$.

Explain why this implies that the physical-state condition is invariant under Lorentz transf., and states appear in complete Lorentz multiplets.

$$\begin{aligned} [L_m, J^\mu] &= \frac{1}{2} \sum_{n=-\infty}^{\infty} (\alpha_{m-n} \cdot \alpha_n, x^\mu p^\mu - \dot{x}^\mu p^\mu - i \sum_{l=1}^{\infty} \frac{1}{l} (\alpha_l^\mu \alpha_{-l}^\mu - \alpha_{-l}^\mu \alpha_l^\mu)) \\ &= \dots - \frac{1}{2} \sum_n \sum_l \frac{1}{l} \left((\alpha_{m-n})^\mu (\alpha_n)_\lambda, \alpha_{-l}^\mu \alpha_l^\lambda - (\alpha_{m-n})^\lambda (\alpha_n)_\mu, \alpha_{-l}^\mu \alpha_l^\lambda \right) \\ &= \dots - \frac{1}{2} \sum_n \sum_l \frac{1}{l} \left((\alpha_{m-n})^\mu [(\alpha_n)_\lambda, \alpha_{-l}^\mu \alpha_l^\lambda] + [(\alpha_{m-n})^\mu, \alpha_{-l}^\mu \alpha_l^\lambda] (\alpha_n)_\lambda - (\alpha_{m-n})^\lambda [(\alpha_n)_\mu, \alpha_{-l}^\mu \alpha_l^\lambda] - [(\alpha_{m-n})^\lambda, \alpha_{-l}^\mu \alpha_l^\lambda] (\alpha_n)_\mu \right) \\ &= \dots - \frac{1}{2} \sum_n \sum_l \frac{1}{l} \left(\underbrace{(\alpha_{m-n})^\mu \alpha_{-l}^\lambda [(\alpha_n)_\lambda, \alpha_l^\mu]}_{n\eta^{\mu\lambda} \delta_{m+l,0}} + \underbrace{(\alpha_{m-n})^\lambda [(\alpha_n)_\mu, \alpha_{-l}^\mu] \alpha_l^\lambda}_{n\eta^{\mu\lambda} \delta_{m+l,0}} + \alpha_{-l}^\lambda [(\alpha_{m-n})^\mu, \alpha_l^\lambda] (\alpha_n)_\lambda + \underbrace{(\alpha_{m-n})^\lambda \alpha_{-l}^\mu \alpha_l^\lambda (\alpha_n)_\mu}_{(m-n)\eta^{\mu\lambda} \delta_{m+n+l,0}} \right. \\ &\quad \left. - (\alpha_{m-n})^\lambda \alpha_{-l}^\mu [(\alpha_n)_\lambda, \alpha_l^\mu] - \underbrace{(\alpha_{m-n})^\lambda (\alpha_n)_\mu \alpha_{-l}^\mu \alpha_l^\lambda}_{n\eta^{\mu\lambda} \delta_{m+l,0}} - \alpha_{-l}^\mu [(\alpha_{m-n})^\lambda, \alpha_l^\mu] (\alpha_n)_\lambda - \underbrace{(\alpha_{m-n})^\mu \alpha_{-l}^\lambda \alpha_l^\lambda (\alpha_n)_\mu}_{(m-n)\eta^{\mu\lambda} \delta_{m+n+l,0}} \right) \\ &= \dots - \frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{l} \left(-\ell \alpha_{m+l}^\mu \alpha_{-l}^\lambda + \ell \alpha_{m-l}^\lambda \alpha_{-l}^\mu - \ell \alpha_{-l}^\mu \alpha_{m+l}^\lambda + \ell \alpha_{-l}^\lambda \alpha_{m-l}^\mu \right. \\ &\quad \left. + \ell \alpha_{m+l}^\lambda \alpha_{-l}^\mu - \ell \alpha_{m-l}^\mu \alpha_{-l}^\lambda + \ell \alpha_{-l}^\mu \alpha_{m+l}^\lambda - \ell \alpha_{-l}^\lambda \alpha_{m-l}^\mu \right) \\ &= \dots - \frac{1}{2} \sum_{l=1}^{\infty} \left(-2\alpha_{m-l}^\lambda \alpha_{m+l}^\mu - \underbrace{(m+l)\eta^{\mu\lambda} \delta_{m,0}}_{m=0 \Rightarrow 0} + 2\alpha_{m-l}^\mu \alpha_{-l}^\lambda + \ell \eta^{\mu\lambda} \delta_{m,0} \right. \\ &\quad \left. + 2\alpha_{m+l}^\lambda \alpha_{-l}^\mu - \ell \eta^{\mu\lambda} \delta_{m,0} - 2\alpha_{-l}^\mu \alpha_{m-l}^\lambda - \underbrace{(m-l)\eta^{\mu\lambda} \delta_{m,0}}_{m=0 \Rightarrow 0} \right) \\ &= \dots + i \sum_{n=1}^{\infty} (\alpha_{-n}^\mu \alpha_{m+n}^\mu - \alpha_{m-n}^\mu \alpha_n^\mu - \alpha_{m+n}^\mu \alpha_{-n}^\mu + \alpha_n^\mu \alpha_{m-n}^\mu) \\ &= \dots + i \sum_{n=0}^{\infty} (\alpha_n^\mu \alpha_{m-n}^\mu - \alpha_{m-n}^\mu \alpha_n^\mu) \\ &= \dots + i \sum_n (\alpha_n^\mu \alpha_{m-n}^\mu - \alpha_{m-n}^\mu \alpha_n^\mu) - i (\alpha_0^\mu \alpha_m^\mu - \alpha_m^\mu \alpha_0^\mu) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_n [(\alpha_{m-n})^\dagger (\alpha_n)_\lambda, \hat{x}^\dagger \hat{p}^\lambda] - [(\alpha_{m-n})^\dagger (\alpha_n)_\lambda, \hat{x} \hat{p}^\lambda] + \dots \\
&= \frac{1}{2} \sum_n (\alpha_{m-n})_\lambda [\alpha_n^\dagger, \hat{x}^\dagger \hat{p}^\lambda] + [\alpha_{m-n}^\dagger, \hat{x} \hat{p}^\lambda] (\alpha_n)_\lambda - (\alpha_{m-n})_\lambda [\alpha_n^\dagger, \hat{x} \hat{p}^\lambda] - [\alpha_{m-n}^\dagger, \hat{x}^\dagger \hat{p}^\lambda] (\alpha_n)_\lambda + \dots \quad \alpha_n^\dagger = \delta_{n,0} \hat{p}^\lambda \\
&= \frac{\alpha_0}{2} (2(\alpha_m)_\lambda [\hat{p}^\lambda, \hat{x}^\dagger \hat{p}^\lambda] - 2(\alpha_m)_\lambda [\hat{p}^\lambda, \hat{x} \hat{p}^\lambda]) + \dots \\
&= -i \delta_0 \alpha_m^\dagger \hat{p}^\lambda + i \delta_0 \alpha_m^\dagger \hat{p}^\lambda + \dots \\
&= -i (\alpha_m^\dagger \alpha_0^\lambda - \alpha_m^\lambda \alpha_0^\dagger) - i (\alpha_0^\lambda \alpha_m^\dagger - \alpha_0^\dagger \alpha_m^\lambda) \\
&= -i (\alpha_m^\dagger \alpha_0^\lambda - \alpha_0^\lambda \alpha_m^\dagger - m \eta^{\mu\nu} \delta_{m,0}) + i (\alpha_0^\lambda \alpha_m^\dagger - \alpha_0^\dagger \alpha_m^\lambda) = 0 //
\end{aligned}$$

Physical state condition $(L_0 - a)|\phi\rangle = 0$, then $(L_0 - a)|\phi\rangle = (L_0 - a)J^{\mu\nu}|\phi\rangle = J^{\mu\nu} \frac{L_0|\phi\rangle}{a|\phi\rangle} - a|\phi\rangle = 0 //$

2.10 Consider an on-shell open-string state of the form $|\phi\rangle = (A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_2 + C(\alpha_0\cdot\alpha_{-1})^2)|0;k\rangle$

Determine the conditions on the coefficients A, B, C st. $|\phi\rangle$ satisfies the physical-state condition

for $a=1$ and arbitrary D. Compute the norm of $|\phi\rangle$. What conclusions can you draw?

L_0 condition:

$$(L_0 - 1)|\phi\rangle = \left(\frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n - 1\right)|\phi\rangle = \left(\frac{1}{2}\alpha_0^2 + 2 - 1\right)|\phi\rangle = \left(\frac{1}{2}\alpha_0^2 + 1\right)|\phi\rangle = 0 \Rightarrow \alpha_0^2 = -2 //$$

-

$$\begin{aligned}
L_n|\phi\rangle &= \frac{1}{2} \sum_{m=0}^{\infty} \alpha_{m-n}\cdot\alpha_m|\phi\rangle = 0 \quad \forall n \neq 0 \\
[L_n, \alpha_2^\dagger] &= \frac{1}{2} \sum_n [(\alpha_{m-n})^\dagger (\alpha_n)_\nu, \alpha_2^\dagger] = \frac{1}{2} \sum_n \left((\alpha_{m-n})_\nu [\alpha_n^\dagger, \alpha_2^\dagger] + [\alpha_{m-n}^\dagger, \alpha_2^\dagger] (\alpha_n)_\nu \right) \\
&\quad \eta^{\mu\nu} \delta_{m+2,0} \quad (m+n)\eta^{\mu\nu} \delta_{m+n+2,0} \\
&= \frac{1}{2} (-2(\alpha_{m+2})^\dagger - 2(\alpha_{m+2})^\dagger) = -2\alpha_{m+2}^\dagger
\end{aligned}$$

L_1 condition

$$\begin{aligned}
L_1(A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_2 + C(\alpha_0\cdot\alpha_{-1})^2)|0;k\rangle &= (A\alpha_{-1}\cdot(L_1\alpha_{-1}) + A\alpha_0\cdot\alpha_{-1} + B\alpha_0\cdot(L_1\alpha_2) + C\alpha_0\cdot(L_1\alpha_{-1})\cdot\alpha_0\cdot\alpha_{-1})|0\rangle \\
&\quad A\alpha_{-1}\cdot(L_1\alpha_{-1}) \quad B\alpha_0\cdot(L_1\alpha_2) \quad C\alpha_0\cdot(L_1\alpha_{-1})\cdot\alpha_0\cdot\alpha_{-1} \\
&\quad -(-1)\alpha_{-1}\cdot A\alpha_{-1} \quad (+0) \quad +(-0) \\
&= (A\alpha_{-1}\cdot\alpha_{-1} L_1 + A\alpha_0\cdot\alpha_0 + A\alpha_0\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_2 L_1 + 2B\alpha_0\cdot\alpha_{-1} + C\alpha_0\cdot\alpha_{-1}\cdot\alpha_0\cdot(L_1\alpha_{-1}) + C\alpha_0^2\alpha_0\cdot\alpha_{-1})|0\rangle \\
&\quad = \alpha_0\cdot\alpha_{-1}|0\rangle \quad \alpha_{-1}L_1 + 2\alpha_{-1} \quad \alpha_{-1}L_1 + 4\alpha_0 \\
&\quad \alpha_{-1}L_1 + 4\alpha_0
\end{aligned}$$

$$\begin{aligned}
\text{Consider } L_1|0;k\rangle &= \frac{1}{2} \sum_n \alpha_{1-n}\cdot\alpha_n|0\rangle = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_{1-n}\cdot\alpha_n|0\rangle + \frac{1}{2} \alpha_{-1}\cdot\alpha_0|0\rangle + \frac{1}{2} \sum_{n=2}^{\infty} \alpha_{1-n}\cdot\alpha_n|0\rangle \\
&\quad \alpha_{-n}\cdot\alpha_{n+1} + (n+1)\eta^{\mu\nu} \delta_{n,0} \quad \alpha_0\cdot\alpha_1 + \eta^{\mu\nu} \delta_{n,0} = 0 \\
&\quad = 0
\end{aligned}$$

$$L_1|\phi\rangle = (2A + 2B - 4C)(\alpha_0\cdot\alpha_{-1})|0;k\rangle \Rightarrow A + B - 2C = 0 //$$

$$L_2|\phi\rangle = (A L_2\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot(L_2\alpha_2) + C\alpha_0\cdot(L_2\alpha_{-1})\cdot\alpha_0\cdot\alpha_{-1})|0\rangle$$

$$\begin{aligned}
&= (A\alpha_{-1}\cdot(\alpha_{-1}L_2 + \alpha_1) + A\alpha_0\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_2L_2 + 2B\alpha_0\cdot\alpha_0 + C\alpha_0\cdot\alpha_{-1}\cdot\alpha_0\cdot(2\alpha_{-1}) + C\alpha_0\cdot\alpha_{-1}\cdot\alpha_0\cdot\alpha_{-1})|0\rangle \\
&= (A\alpha_{-1}\cdot\alpha_{-1} + A\alpha_0\cdot\alpha_{-1} - 4B - 2C)|0\rangle \\
&\quad \alpha_{-1}\cdot\alpha_{-1} + (n)\eta^{\mu\nu} \delta_{n,0} \quad C\alpha_0\cdot\alpha_{-1}\cdot\alpha_0 \\
&\quad = \dots + C(1)\delta_{n,0}\alpha_0^2
\end{aligned}$$

note: $[L_n, \alpha_{-2}^\dagger] = +2\alpha_{n-2}^\dagger$
 $\Rightarrow n=2$, automatically zero on $|0\rangle$

$$\Rightarrow AD - 4B - 2C = 0, A + B - 2C = 0 //$$

$$AD - A - 4B - B = 0 \Rightarrow B = \frac{A}{5}(D-1)$$

$$2C = A + B \Rightarrow \frac{A}{10}(D+4)$$

2.11 The open-string states at the $N=2$ level were shown in section 2.5 to form certain reps of $SO(25)$.

What does this imply for the closed string at the $N_L=N_R=2$ level?

massive since $M^2 = \frac{\alpha'}{4} (N-1) \rightarrow SO(25)$

For a closed-string at $N_L=N_R=2$, we have left and right moving parts independently at the whole closed string is in reps of $SO(25) \times SO(25)$

Consider left moving modes at $N=2$,

$$\underbrace{\alpha_{-2}^i, \alpha_{-1}^i, \alpha_{-1}^i}_{24 + \frac{25 \times 24}{2} = 324} \longrightarrow 324 \text{ states} = \dim \text{Sym}^2([0 \dots 0]_{25}) - [0 \dots 0] = \frac{25 \times 25}{2} - 1 = 324$$

form the rep $[20 \dots 0]_{25}$

Full spectrum is $[20 \dots 0][20 \dots 0]$

2.12 Construct the spectrum of open and closed strings in light-cone gauge for $N=3$.

How many states are there in each case? Describe a strategy for assembling the states into reps of $SO(25)$.

For $N=3$: $\alpha_{-3}^i |0; k\rangle, \alpha_{-2}^i \alpha_{-1}^j |0; k\rangle, \alpha_{-1}^i \alpha_{-1}^j \alpha_{-1}^k |0; k\rangle$

$$\begin{aligned} & 24 \quad 24 \times 24 = 576 \quad \frac{24 \times 25 \times 24}{6} = 2600 = 3200 \\ & = \text{Sym}^3([0 \dots 0]_{25}) - [0 \dots 0]_{25} + R([0 \dots 0]_{25}) \\ & \left(\frac{25 \times 24}{2} \right) = \frac{27 \times 26 \times 25}{6} = 2925 \quad \left(\frac{25}{2} \right) = 300 = 3300 \end{aligned}$$

trace for rank-2: $\sum 2\delta_{ik} = 0 \quad \forall k=1 \dots 25$

2.13 We expect the central extension of the Virasoro algebra to be of the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}$$

because normal ordering ambiguities only arise for $m+n=0$.

(i) Show that if $A(n) \neq 0$ it is possible to change the definition of L_0 by adding a constant,

so that $A(n)=0$.

(ii) For $A(n)=0$, show that $L_0, L_{\pm 1}$ form a subalgebra.

Consider $[L_1, L_{-1}] = 2L_0 + A(1)$, let $\tilde{L}_0 = L_0 + \frac{1}{2}A(1)$ then $[L_1, L_{-1}] = 2\tilde{L}_0 \longrightarrow \mathfrak{sl}(2)$ algebra

2.14 Derive an equation for $A(n)$ from the Jacobi identity.

Assuming $A(n)=0$, prove that $A(n) = \frac{(n^2-n)A(2)}{6}$ is the unique solution and hence that the central charge is $28\alpha'$.

Jacobi: $[[L_m, L_n], L_p] + [[L_n, L_p], L_m] + [[L_p, L_m], L_n] = 0$

$$= (m-n)[L_{m+n}, L_p] + (n-p)[L_{n+p}, L_m] + (p-m)[L_{p+m}, L_n]$$

$$= (m-n)(m+n-p)L_{m+n+p} + (m-n)A(m+n)S_{m+n,p,0} + (n-p)(n+p-m)L_{n+p+m} + (n-p)A(n+p)S_{n+p,m,0} + (p-m)(p+m-n)L_{p+m+n} + (p-m)A(p+m)S_{p+m,n,0}$$

$$= L_{mnp} \left((m-n)(m+n-p) + (n-p)(n+p-m) + (p-m)(p+m-n) \right) + \underbrace{\delta_{mnp,0}}_{\substack{\Rightarrow m+n+p=0 \\ p=m-n}} \left((m-n)A(m+n) + (n-p)A(n+p) + (p-m)A(p+m) \right) = 0$$

$$\Rightarrow L_0 \left(\underbrace{2(m-n)(m+n)}_{2m^2-2n^2} + \underbrace{(2n+m)(-2m)}_{-4mn-2m^2} + \underbrace{(-2m-n)(-2n)}_{4mn+2n^2} \right) + (m-n)A(m+n) + (2n+m)A(-m) + (-2m-n)A(-n) = 0 //$$

Assuming $A(n)=0$, let $n=1-m$ st. $\underbrace{(m-(1-m))A(n)}_{=0} + (2-m)A(-m) + (-m-1)A(m-1) = 0$

$$(2-m)A(-m) = (m+1)A(m-1)$$

$$\uparrow \quad (2-m)A(-m) = (m+1)A(m-1)$$

$$\Rightarrow (2-0)A(-1) = (1+1)A(0) \Rightarrow A(0) = 0$$

For $n=-1$, $(m+1)A(m-1) + (m+2)A(-m) + \underbrace{(1-2m)A(1)}_0 = 0$

$$\Rightarrow 2A(0) = A(-1) \Rightarrow A(-1) = 0$$

For $n=1$, $(m-1)A(m+1) + (m+2)A(-m) - \underbrace{(2m+1)A(-1)}_0 = 0$

$$\Rightarrow (1-m)A(m+1) = (m+2) \frac{(m+1)A(m-1)}{(2-m)}$$

$$\Rightarrow A(m+1) = \frac{(m+1)(m+2)A(m-1)}{(1-m)(2-m)}$$

$$A(m) = \frac{m(m+1)A(m-2)}{(2-m)(3-m)} = \frac{m(m+1)}{(2-m)(3-m)} \cdot \frac{(m+2)(m-1)}{(4-m)(5-m)} \cdot \frac{(-1)(m+3)(m-3)}{(6-m)(7-m)} A(m-6)$$

$$= m(m+1)(m-1) \frac{A(m)}{6} = m(m-1) \frac{A(2)}{6} //$$

$$\dots \frac{(3-6)}{6} \cdot \frac{(2-4)(3-4)}{-3} \cdot \frac{4(8)}{4} A(2)$$

 Bouchard.

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