

3.1 Compute the commutator of an infinitesimal translation and a special conformal transformation.

Identify the resulting transformation.

Conformal sym. preserving: translations, rotations (incl. Lorentz), scale, inversion, SCT (inversion-translation-inversion)

$dx \rightarrow e^{i\omega} dx$
 $x^\mu \rightarrow x^\mu + \epsilon^\mu$ $x^\mu \rightarrow \omega^\mu_\nu x^\nu$ $x^\mu \rightarrow \lambda x^\mu$ $x^\mu \rightarrow \frac{x^\mu}{x^2}$

Let SCT: $x^\mu \rightarrow \frac{x^\mu}{x^2} \rightarrow \frac{x^\mu}{x^2} + b^\mu \rightarrow \frac{\frac{x^\mu}{x^2} + b^\mu}{\frac{x^2}{x^2} + 2\frac{x^\mu b_\mu}{x^2} + b^2} = \frac{x^\mu + x^2 b^\mu}{1 + 2x^\mu b_\mu + x^2 b^2} = (x^\mu + x^2 b^\mu)(1 - 2x^\mu b_\mu + \dots) = x^\mu + \underbrace{x^2 b^\mu - 2x^\mu x^\nu b_\nu}_{\delta x^\mu} + \dots$

translation
 $\uparrow$
 $[S_0, S_a] x^\mu = (\delta_0 \delta_a - \delta_a \delta_0) x^\mu = \delta_a (b^\mu x^\mu - 2x^\mu x^\nu b_\nu) - \delta_0 (a^\mu)$

$$= 2b^\mu x^\nu \delta_{\nu\mu} - 2(\delta_a x^\mu)(a \cdot b) - 2x^\mu (\delta_a \cdot b)$$

$$= 2b^\mu (a \cdot x) - 2a^\mu (b \cdot x) - 2x^\mu (a \cdot b) = \delta_a x^\mu + \delta_a x^\mu$$

$$= 2(\delta_a \nu - a^\mu \delta_\nu) x^\mu = \lambda x^\mu$$

SCALE

3.2 Show that the transformations (3.14) give rise to the D=2 case of the D-dimensional transformations in Eq. (3.7).

Transformations (3.14):
 $L_1: z \rightarrow z - \bar{z}$
 $L_0: z \rightarrow z - z\bar{z}$
 $L_1: z \rightarrow z - z\bar{z}$

Transformations in (3.7): $Sx^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b \cdot x)$

2d $\left(\begin{pmatrix} Sx^0 \\ Sx^1 \end{pmatrix} = \begin{pmatrix} a^0 \\ a^1 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} + \lambda \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} + \begin{pmatrix} b^0 \\ b^1 \end{pmatrix} (x^0 + x^1)^2 - 2 \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} (b^0 + b^1 x) \right)$

$$\begin{aligned} \delta x^0 + i \delta x^1 &= (a^0 + \omega x^1 + \lambda x^0 + b^0 (x^0 + x^1)^2 - 2x^0 (b^0 + b^1 x)) \\ &\quad + i(a^1 - \omega x^0 + \lambda x^1 + b^1 (x^0 + x^1)^2 - 2x^1 (b^0 + b^1 x)) \\ &= (a^0 + i a^1) + \omega (x^1 - i x^0) + (b^0 + i b^1) (x^0 + x^1)^2 - 2(x^0 + i x^1) (b^0 + i b^1 x) \\ &= \underbrace{a^0 + i a^1}_{\in \mathbb{C}} + \underbrace{\omega (x^1 - i x^0)}_{-2i\omega \frac{z^2 + \bar{z}^2}{2}} + \underbrace{(b^0 + i b^1)}_{\in \mathbb{C}} (x^0 + x^1)^2 - 2 \underbrace{(x^0 + i x^1)}_{\in \mathbb{C}} (b^0 + i b^1 x) \end{aligned}$$

$\delta z = a + \beta z + \gamma \bar{z}$

$\Rightarrow \delta z = \delta x + i \delta y = a + \beta (x + iy) + \gamma (x - iy)$

3.3 Show that the algebra of Lorentzian-signature conformal transformations in D dimensions is isomorphic to the Lie algebra $so(0,2)$.

For $so(0,2)$, we have the standard result: $[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}$

Since $\eta_{\mu\nu} = \text{diag}(\underbrace{-, \dots, -}_{\alpha}, \underbrace{+}_{\beta})$:

$$\begin{aligned} [M_{ij}, M_{mn}] &= \eta_{im} M_{jn} - \eta_{jm} M_{in} - \eta_{in} M_{jm} - \eta_{jn} M_{im} \\ [M_{\alpha\beta}, M_{\alpha\gamma}] &= \eta_{\alpha\gamma} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\gamma} - \eta_{\alpha\gamma} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\gamma} \\ [M_{\alpha\alpha}, M_{ij}] &= -\eta_{ij} M_{\alpha\alpha} - \eta_{\alpha\alpha} M_{ij} \\ [M_{ij}, M_{\alpha\beta}] &= 0 \\ [M_{ij}, M_{mn}] &= \eta_{jm} M_{in} - \eta_{im} M_{jn} \\ [M_{\alpha\beta}, M_{i\gamma}] &= \eta_{\alpha\gamma} M_{\beta i} - \eta_{\beta\gamma} M_{\alpha i} \end{aligned}$$

For conformal transformations, $Sz^{\mu} = \frac{a^{\mu} + b^{\mu} \cdot z^{\nu}}{c^{\mu} + d^{\mu} \cdot z^{\nu}}$
 Generators: P_{μ} $N_{\mu\nu}$ D K_{μ}
 M_{0i} M_{ij} M_{0i} M_{ij} Guess

$$M_{\mu\nu} = \begin{pmatrix} 0 & 1 & 2 & \dots & D-1 \\ 0 & M_{01} & M_{02} & \dots & M_{0,D-1} \\ 0 & M_{10} & M_{11} & \dots & M_{1,D-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & M_{D-1,0} & M_{D-1,1} & \dots & M_{D-1,D-1} \end{pmatrix}$$

$$[N^{\mu\nu}, N^{\rho\sigma}] = \eta^{\mu\rho} N^{\nu\sigma} - \eta^{\nu\rho} N^{\mu\sigma} - \eta^{\mu\sigma} N^{\nu\rho} + \eta^{\nu\sigma} N^{\mu\rho} \longleftrightarrow [M_{ij}, M^{kl}] = \dots \checkmark$$

$$[P^{\mu}, K^{\nu}] = 2N^{\mu\nu} - 2D\eta^{\mu\nu} \text{ (see problem 3.1)} \longleftrightarrow [M_{0i}, M_{ij}] = -\eta_{ij} M_{0i} \quad \times$$

$$[M_{0i} + M_{ii}, M_{0j} - M_{ij}] = [M_{0i}, M_{0j}] - [M_{0i}, M_{ij}] + [M_{ii}, M_{0j}] - [M_{ii}, M_{ij}]$$

$$= -\eta_{00} M_{ij} - \eta_{ij} M_{00} + \eta_{ij} M_{0i} - \eta_{ij} M_{i0} - \eta_{ii} M_{0j} - \eta_{ij} M_{jj}$$

$$= 2M_{ij} - 2\eta_{ij} M_{00} \quad \checkmark$$

etc... $i=2, \dots, D-1$
 $\mu=0, \dots, D-1$

$$\Rightarrow M_{ij} = N_{\mu\nu}, \quad M_{0i} + M_{ii} = P_{\mu}, \quad M_{0i} - M_{ii} = K_{\mu}, \quad M_{00} = D \Rightarrow \text{conf}(D) = \text{so}(D, 2)$$

3.4) Define the OPE $T(z)\chi^{\dagger}(w, \bar{w}) \sim \frac{1}{z-w} \partial\chi^{\dagger}(w, \bar{w}) + \dots$

What does this imply for the conformal dimension?

ASIDE:

The energy momentum tensor is $T_{\mu\nu} = \partial_{\mu}X \cdot \partial_{\nu}X - \frac{1}{2}\eta_{\mu\nu} \eta^{\rho\sigma} \partial_{\rho}X \cdot \partial_{\sigma}X$

$$\Rightarrow \partial^{\mu}T_{\mu\nu} = (\partial^{\mu}\partial_{\mu}X) \cdot (\partial_{\nu}X) + (\partial_{\nu}X) \cdot (\partial^{\mu}\partial_{\mu}X) - \frac{1}{2}\eta^{\mu\sigma}(\partial_{\mu}\partial_{\sigma}X) \cdot (\partial_{\nu}X) - \frac{1}{2}\eta^{\mu\sigma}(\partial_{\sigma}X) \cdot (\partial_{\mu}\partial_{\nu}X)$$

$$= \partial_{\nu}X \cdot \partial^{\mu}\partial_{\mu}X + \partial_{\mu}X \cdot \partial_{\nu}\partial^{\mu}X + \frac{1}{2}(\partial_{\mu}\partial_{\nu}X) \cdot (\partial^{\mu}X) - \frac{1}{2}(\partial_{\mu}\partial^{\mu}X) \cdot (\partial_{\nu}X)$$

$$+ \frac{1}{2}(\partial_{\nu}X) \cdot (\partial_{\mu}\partial^{\mu}X) - \frac{1}{2}(\partial_{\mu}X) \cdot (\partial_{\nu}\partial^{\mu}X) = 0$$

$$\Rightarrow T^{\mu}_{\mu} = \eta^{\mu\nu}T_{\mu\nu} = \partial_{\mu}X \cdot \partial^{\mu}X - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu} \partial_{\rho}X \cdot \partial^{\rho}X = 0$$

$-T_{11} + T_{22}$ $\frac{D^2-1}{2}$

Doing a Wick rotation: $\tau \rightarrow i\tau_e \Rightarrow ds^2 = -dt^2 + d\vec{x}^2 = -(i d\tau_e)^2 + d\vec{x}^2 = d\tau_e^2 + d\vec{x}^2$

$$\partial^{\mu}T_{\mu\nu} = \partial^{\mu}(\partial_{\mu}X \cdot \partial_{\nu}X - \frac{1}{2}\delta_{\mu\nu} \partial_{\rho}X \cdot \partial^{\rho}X) = 0 \quad \text{same as above} \quad \partial_{\mu}T_{\mu\nu} = 0 \Rightarrow \partial_1 T_{11} + \partial_2 T_{21} = 0$$

$$(\tau^{\mu}_{\nu})^{\rho}_{\sigma} = \delta^{\mu\rho} \delta^{\sigma}_{\nu} = \partial_{\mu}X \cdot \partial^{\rho}X - \frac{1}{2}\delta^{\mu\rho} \delta_{\nu\sigma} \partial_{\alpha}X \cdot \partial^{\alpha}X = 0$$

$T_{11} + T_{22}$ $\frac{D^2-1}{2}$

Let $z = \tau_e + i\sigma$, $\bar{z} = \tau_e - i\sigma$
 $\tau_e = \frac{1}{2}(z + \bar{z})$, $\sigma = \frac{i}{2}(z - \bar{z})$

$$\Rightarrow T_{z\bar{z}} = \frac{\partial\sigma}{\partial z} \frac{\partial\sigma}{\partial \bar{z}} T_{\sigma\sigma} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)T_{\sigma\sigma} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)T_{\tau\tau} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)T_{\tau\sigma} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)T_{\sigma\tau}$$

$$= \frac{1}{4}(T_{\tau\tau} + T_{\sigma\sigma}) = 0$$

$$\Rightarrow \partial T_{z\bar{z}} = \left(\frac{\partial\sigma}{\partial z}\right) \partial \left(\left(\frac{\partial\sigma}{\partial \bar{z}}\right) T_{\sigma\sigma} \right)$$

$$= \left(\frac{1}{2}\partial_1 + \frac{i}{2}\partial_2\right) \left(\frac{1}{2}\frac{1}{2}T_{\sigma\sigma} + \frac{1}{2}\frac{1}{2}T_{\tau\tau} + \frac{1}{2}\frac{1}{2}T_{\tau\sigma} + \frac{1}{2}\frac{1}{2}T_{\sigma\tau} \right)$$

$$= \frac{1}{8}(\partial_1 + i\partial_2)(T_{\sigma\sigma} - 2iT_{\tau\sigma} - T_{\tau\tau})$$

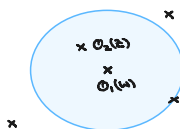
$$= \frac{1}{8}(\partial_1 T_{\sigma\sigma} - \partial_1 T_{\tau\tau} + 2\partial_2 T_{\sigma\sigma}) + \frac{1}{8}(-2\partial_2 T_{\tau\sigma} + \partial_2 T_{\sigma\sigma} - \partial_2 T_{\tau\tau})$$

$$= \frac{1}{4}(\partial_1 T_{\sigma\sigma} + \partial_2 T_{\tau\tau}) - \frac{i}{4}(\partial_1 T_{\tau\sigma} + \partial_2 T_{\sigma\tau}) = 0$$

Operator product expansion (OPE):
 exact w/ radius of convergence

$$\mathcal{O}_i(z, \bar{z}) \mathcal{O}_j(w, \bar{w}) = \sum_k C_{ij}^k(z, w, \bar{z}, \bar{w}) \mathcal{O}_k(w, \bar{w})$$

statement about operator, thus we really mean
 time ordered $\langle \mathcal{O}_i \mathcal{O}_j \dots \rangle = \sum_k C_{ij}^k \langle \mathcal{O}_k \dots \rangle$



$$T_{33} = \frac{\partial \mathcal{L}}{\partial z} \frac{\partial \mathcal{L}}{\partial \bar{z}} T_{33} = \frac{1}{4} (T_{11}^2 - 2T_{12}T_{21} - T_{22}^2) = \frac{1}{4} (2X \partial X - \frac{1}{2} (2X \partial X + 2\bar{X} \partial \bar{X}) - 2i \partial X \partial \bar{X} - 2\bar{X} \partial X + \frac{1}{2} (2X \partial X + 2\bar{X} \partial \bar{X}))$$

$$\partial X \partial X = \frac{1}{2} (\partial_1 - i\partial_2) X \cdot \frac{1}{2} (\partial_1 - i\partial_2) X = \frac{1}{4} (\partial_1 X \partial_1 X - \partial_1 X \partial_2 X - 2i \partial_1 X \partial_2 X)$$

MORE ASIDE: TONG 9.2.2

GAMES W/ PATH INTEGRALS *or fields*

Schematically: $Z = \int D\phi e^{-S[\phi]}$

A symmetry is such that: $\phi \rightarrow \phi + \epsilon \delta\phi$ leaves both action & measure invariant: $S[\phi] = S[\phi + \epsilon \delta\phi]$ & $D\phi = D(\phi + \epsilon \delta\phi)$ *technically only their combination*

Promote $\epsilon \rightarrow \epsilon(x)$, then $\partial_\mu(\phi) = \partial_\mu(\phi + \epsilon \delta\phi) = \partial_\mu \phi + \epsilon \partial_\mu \delta\phi + (\partial_\mu \epsilon) \delta\phi$

$$\epsilon \rightarrow \int D\phi e^{-S[\phi]} = \int D(\phi + \epsilon \delta\phi) e^{-S[\phi + \epsilon \delta\phi]}$$

$$D\phi = D(\phi + \epsilon \delta\phi) = e^{\epsilon^A} D\phi$$

$$\delta S = \frac{\partial S}{\partial \phi} \delta\phi + \frac{\partial S}{\partial \partial_\mu \phi} \delta(\partial_\mu \phi)$$

$$= \epsilon \left(\frac{\partial S}{\partial \phi} \delta\phi + \frac{\partial S}{\partial \partial_\mu \phi} \delta(\partial_\mu \phi) \right)$$

$$= \epsilon \left(\frac{\partial S}{\partial \phi} \delta\phi + \frac{\partial S}{\partial \partial_\mu \phi} \delta(\partial_\mu \phi) \right) + \frac{\partial S}{\partial \partial_\mu \phi} \delta\phi (\partial_\mu \epsilon)$$

original symmetry, so for $\delta\epsilon = 0$, this is $\epsilon \partial_\mu T^{\mu\nu}$

$$\delta S = \int d^d x \delta\phi \delta\pi$$

$$= \int d^d x \delta\phi \left(\epsilon \partial_\mu T^{\mu\nu} + (\partial_\mu \epsilon) \frac{\partial S}{\partial \partial_\mu \phi} \delta\phi \right)$$

$$= \int d^d x \delta\phi \left(\partial_\mu \epsilon \right) \left(-T^{\mu\nu} + \frac{\partial S}{\partial \partial_\mu \phi} \delta\phi \right)$$

$$= \partial_\mu \frac{1}{2\pi} \text{ Noether current}$$

$$= \int D\phi e^{\epsilon^A} \partial_\mu \left(-S[\phi] \right) - \frac{1}{2\pi} \int d^d x \partial_\mu \epsilon = \int D\phi e^{-S[\phi]} e^{-\frac{1}{2\pi} \int d^d x \partial_\mu \epsilon + \text{tr } A}$$

$$= \int D\phi e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int d^d x \partial_\mu \epsilon \right) = Z$$

$-\frac{1}{2\pi} \int d^d x \partial_\mu \epsilon$ *now includes contribution from S & D\phi.*

$$\Rightarrow \int D\phi e^{-S[\phi]} \left(\int d^d x \partial_\mu \epsilon \right) = 0 \longrightarrow \langle \partial_\mu T^{\mu\nu} \rangle = 0$$

$$\begin{matrix} \phi_1 & & \phi_2 \\ \times & & \times \\ & \phi_3 & \times \phi_4 \end{matrix}$$

Time ordered correlation function for various operators: $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int D\phi e^{-S[\phi]} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)$

Under the symmetry, $\mathcal{O}_1 \rightarrow \mathcal{O}_1 + \epsilon \delta\mathcal{O}_1$

Choose $\epsilon(x)$ st. $\mathcal{O}_1 \times \text{red circle} \times \mathcal{O}_2$ then $\mathcal{O}_1 \rightarrow \mathcal{O}_1$, the same way as above... $\int D\phi e^{-S[\phi]} \left(\int d^d x \partial_\mu \epsilon \right) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) = 0$

$\Rightarrow \langle \partial_\mu T^{\mu\nu}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = 0$

$\hookrightarrow$ write this as $\partial_\mu T^{\mu\nu} = 0$

Now consider $\mathcal{O}_1 \times \text{red circle} \times \mathcal{O}_2$ then $\mathcal{O}_1 \rightarrow \mathcal{O}_1 + \epsilon \delta\mathcal{O}_1$, the correlation function $\frac{1}{Z} \int D\phi e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int d^d x \partial_\mu \epsilon \right) (\mathcal{O}_1 + \epsilon \delta\mathcal{O}_1) \mathcal{O}_2 \dots \mathcal{O}_n$

$\Rightarrow \frac{1}{Z} \int D\phi e^{-S[\phi]} \left(\int d^d x \partial_\mu (T^{\mu\nu} \dots \mathcal{O}_n) \right) + \frac{1}{Z} \int D\phi e^{-S[\phi]} \epsilon (\delta\mathcal{O}_1) \mathcal{O}_2 \dots \mathcal{O}_n = 0$

$\Rightarrow \frac{1}{2\pi} \int d^d x \langle \partial_\mu (T^{\mu\nu} \dots \mathcal{O}_n) \rangle = - \langle \delta\mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle$ **WARD IDENTITY**

WARD IDENTITIES FOR CONFORMAL TRANSFORMATIONS

In 2d, we can use Stokes' theorem. $(ds^\mu, ds^\nu) \rightarrow \text{if } ds^\mu$

$$\int_\Sigma ds^\mu \partial_\mu T^{\mu\nu} = \oint_{\partial\Sigma} ds^\mu \bar{\partial}_\mu T^{\mu\nu}$$

$$= \oint_{\partial\Sigma} (J_1 ds^2 - J_2 d\bar{s})$$

$$= \oint_{\partial\Sigma} (J_2 + J_3) \left(\frac{1}{2} (dz - d\bar{z}) \right) - i(J_2 - J_3) \left(\frac{1}{2} (dz + d\bar{z}) \right)$$

$$J_2 = \frac{1}{2} (J_1 - iJ_2) \neq J_3 = \frac{1}{2} (J_1 + iJ_2)$$

$$J_1 = J_2 + J_3 \quad iJ_2 = J_3 - J_2 \quad J_2 = i(J_3 - iJ_2)$$

$$\left\{ \begin{array}{ll} z = t + i\sigma & \bar{z} = t - i\sigma \\ t = \frac{1}{2}(z + \bar{z}) & i\sigma = \frac{1}{2}(z - \bar{z}) \\ dt = \frac{1}{2}(dz + d\bar{z}) & d\sigma = \frac{1}{2i}(dz - d\bar{z}) \end{array} \right.$$

$$= -\frac{i}{2} \oint_{\partial\Sigma} J_2 dz - J_2 d\bar{z} + J_3 dz - J_3 d\bar{z} + J_2 dz + J_3 d\bar{z} - J_2 d\bar{z} - J_3 d\bar{z} = -i \oint_{\partial\Sigma} J_2 dz - J_3 d\bar{z}$$

The Ward identity becomes: $\frac{i}{2\pi} \oint dz \langle J_3(z, \bar{z}) O_1(\sigma_1) \dots \rangle - \frac{i}{2\pi} \oint d\bar{z} \langle J_3(z, \bar{z}) O_1(\sigma_1) \dots \rangle = \langle \underbrace{\bar{\partial} O_1(\sigma_1)}_{(\bar{\partial}_2 O_1 + \bar{\partial}_1 O_1)} O_2(\sigma_2) \dots \rangle$

$\partial_z \bar{z} = \partial_z z + \bar{\partial}_z \bar{z} = 0$
 $\Rightarrow J_3$ holomorphic
 J_3 anti-holomorphic
 $\rightarrow$ Residue theorem works

And thus $\frac{i}{2\pi} \oint dz J_3(z) O_1(\sigma_1) = -\text{Res}(J_3 O_1)$

$$\hookrightarrow J_3(z) O_1(w, \bar{w}) = \dots + \frac{\text{Res}(J_3 O_1(w, \bar{w}))}{z-w} + \text{regular terms}$$

For the above

$$\langle \bar{\partial}_2 O_1(\sigma_1) \rangle = \frac{i}{2\pi} \oint dz \langle J_3(z) O_1(\sigma_1) \rangle = -\text{Res}[J_3(z) O_1(\sigma_1)] = -\text{Res}[\underbrace{\bar{\partial}_2 O_1}_{\text{anomalous current}}(z) O_1(\sigma_1)]$$

$$\langle \bar{\partial}_1 O_1(\sigma_1) \rangle = -\frac{i}{2\pi} \oint d\bar{z} \langle J_3(z, \bar{z}) O_1(\sigma_1) \rangle = -\frac{i}{2\pi} \oint d\bar{z} \langle J_3(z, \bar{z}) O_1(\sigma_1) \rangle = -\text{Res}[\bar{\partial}_1 O_1(z, \bar{z}) O_1(\sigma_1)]$$

Def An operator O is said to have weight $(h, \bar{h})$ if under $Sz = z\bar{z}$ and $S\bar{z} = \bar{z}z$, O transforms as

$$SO = -z(hO + \bar{z}\partial O) - \bar{z}(\bar{h}O + z\partial O)$$

Classical: $T(z) = -\frac{1}{\alpha'} \partial X \partial X$

vertex/progenerator / 2pt corr.

Quantum: $T(z) = -\frac{1}{\alpha'} : \partial X \partial X : = -\frac{1}{\alpha'} \lim_{z \rightarrow w} \left(\partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle \right) \Rightarrow \langle T \rangle = 0$

From $S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\mu X \partial^\mu X$ w/ eom $\partial^2 X = 0$,

$$0 = \int dX \frac{\delta}{\delta X(\sigma)} e^S = \int dX e^S \left(-\frac{\delta S}{\delta X} \right) = \int dX e^S \left(-\frac{2}{4\pi\alpha'} \partial^2 X \right) \rightarrow \langle \partial^2 X \rangle = 0$$

Emphasize that: the exp. value of operators must obey the classical eom.

For the propagator, we consider

$$0 = \int dX \frac{\delta}{\delta X(\sigma)} \left(e^S X(\sigma') \right) = \int dX e^S \left(\frac{1}{2\pi\alpha'} (\partial^2 X(\sigma') X(\sigma') + S(\sigma-\sigma')) \right) \rightarrow \langle \partial^2 X(\sigma') X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma-\sigma')$$

$$\partial_\sigma \langle X(\sigma') X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma-\sigma')$$

Using the standard result: $\partial^2 \ln(\sigma-\sigma')^2 = 4\pi \delta(\sigma-\sigma')$

$$\partial_\sigma^2 \langle X(\sigma') X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma-\sigma')$$

$$\partial_\sigma^2 (\ln(\sigma-\sigma')^2) = 4\pi \delta(\sigma-\sigma')$$

$$\Rightarrow \langle X(\sigma') X(\sigma') \rangle = -\frac{\alpha'}{2} \ln(\sigma-\sigma')^2$$

$$X(\sigma') X(\sigma') = -\frac{\alpha'}{2} \ln(\sigma-\sigma')^2 + \dots$$

In $\mathbb{C}$, we have the eom $\partial \bar{\partial} X = 0 \Rightarrow X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$

$$\text{and } (\sigma-\sigma')^2 = |z-w|^2 = (z-w)(\bar{z}-\bar{w})$$

$$X(z)X(w) + \bar{X}(\bar{z})\bar{X}(\bar{w}) + \text{mixed} = -\frac{\alpha'}{2} \ln((z-w)(\bar{z}-\bar{w})) = -\frac{\alpha'}{2} \ln(z-w) - \frac{\alpha'}{2} \ln(\bar{z}-\bar{w})$$

$$\Rightarrow X(z)X(w) = -\frac{\alpha'}{2} \ln(z-w)$$

$$\text{Then } \langle \partial_z X(z) \partial_w X(w) \rangle = \partial_z \partial_w \langle X(z)X(w) \rangle = -\frac{\alpha'}{2} \partial_z \partial_w \ln(z-w) = -\frac{\alpha'}{2} \partial_z \frac{-1}{z-w} = -\frac{\alpha'}{2} \frac{1}{(z-w)^2}$$

same ordered, as we take it inside time ordered corr. func.

$$T(\partial X \partial X) = \left(-\frac{1}{\alpha'} : \partial X(z) \partial X(w) : \right) \partial X(w)$$

$$= -\frac{1}{\alpha'} : \partial X(z) \partial X(w) \partial X(w) : = -\frac{1}{\alpha'} \partial X(z) \langle \partial X(z) \partial X(w) \rangle = -\frac{1}{\alpha'} \partial X(z) \left(-\frac{\alpha'}{2} \frac{1}{(z-w)^2} \right)$$

which non-singular by definition

$$f(w) = f(z) + \alpha \partial f(z) + \dots$$

$$\frac{\partial f(z)}{\partial w} = \frac{\partial f(z)}{\partial z} \frac{\partial z}{\partial w} + \dots = \frac{\partial f(z)}{\partial z} \frac{1}{z-w} + \frac{\partial^2 f(z)}{\partial z^2} \frac{1}{(z-w)^2} + \dots$$

finally...

$$T(z)X(u, \bar{u}) = \left(-\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \right) (X(u) + \bar{X}(u)) = -\frac{2}{\alpha'} \partial X(z) \langle \partial X(z) X(u) \rangle + \dots = \frac{\partial X(z)}{z-u} + \dots = \frac{\partial X(u)}{z-u} + \partial^2 X(u) + \dots$$

non-singular

$$= \partial_z \langle X(z) X(u) \rangle = -\frac{2}{\alpha'} \partial_z \ln(z-u) = -\frac{2}{\alpha'} \frac{1}{z-u}$$

$$\delta X(u, \bar{u}) = -\text{Res} [\partial_z X(u)] = -\text{Res} \left[\frac{2}{z-u} X(u) \right] = 0, \quad T(z) \partial(u, \bar{u}) = \dots + \ln \frac{\partial(u, \bar{u})}{(z-u)^2} + \frac{\partial \partial(u, \bar{u})}{(z-u)} + \dots$$

(z-u) term ...

⇒ For $X(u, \bar{u})$, we have $(h, \bar{h}) = (0, 0)$

(3.5) i) Use the results of the previous problem to determine the CPE of $T(z)$ with: $\partial X^{\mu}(u, \bar{u})$, $\bar{\partial} X^{\mu}(u, \bar{u})$, $\partial^2 X^{\mu}(u, \bar{u})$

$\partial(X(u) + \bar{X}(u)) = \partial X(u)$

$$T(z) \partial X(u, \bar{u}) = \left(-\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \right) \partial X(u) = -\frac{2}{\alpha'} \partial X(z) \overline{\partial X(z) \partial X(u)} + \dots$$

$$= -\frac{2}{\alpha'} \partial X(z) \left(-\frac{\alpha'}{2} \frac{1}{(z-u)^2} \right) + \dots = \frac{\partial X(z)}{(z-u)^2} + \dots = \frac{\partial X(u)}{(z-u)^2} + \frac{\partial^2 X(u)}{(z-u)} + \dots$$

$f(u+v) = f(u) + v \partial f(u) + \dots$
 $a = z-u \rightarrow f(z) = f(u) + (z-u) \partial f(u) + \dots$

$$T(z) \bar{\partial} X(u, \bar{u}) = \left(-\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \right) \bar{\partial} X(u) = \dots$$

↳ contraction is zero

$$T(z) \partial^2 X(u, \bar{u}) = \left(-\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \right) \partial^2 X(u) = -\frac{2}{\alpha'} \partial X(z) \langle \partial X(z) \partial^2 X(u) \rangle + \dots = 2 \frac{\partial X(z)}{(z-u)^3} + \dots = 2 \frac{\partial X(u)}{(z-u)^3} + 2 \frac{\partial^2 X(u)}{(z-u)^2} + 2 \frac{\partial^3 X(u)}{(z-u)} + \dots$$

$\partial_u \langle \partial X(z) \partial X(u) \rangle = \partial_u \left(-\frac{\alpha'}{2} \frac{1}{(z-u)^2} \right) = \alpha' \frac{1}{(z-u)^3}$

(3.5) ii) What do these results imply for the conformal dimension $(h, \bar{h})$ in each case?

$$\partial X(u, \bar{u}) = \partial X(u) : (h, \bar{h}) = (1, 0)$$

$$\bar{\partial} X(u, \bar{u}) = \bar{\partial} X(u) : (h, \bar{h}) = (0, 1)$$

$$\partial^2 X(u, \bar{u}) = \partial^2 X(u) : (h, \bar{h}) = (2, 0)$$

(3.6) Show that $[\alpha_n^{\mu}, \alpha_n^{\nu}] = [\bar{\alpha}_n^{\mu}, \bar{\alpha}_n^{\nu}] = i\eta^{\mu\nu} \delta_{n,0}$ & $[\alpha_n^{\mu}, \bar{\alpha}_n^{\nu}] = 0$ by using the CPE of $\partial X^{\mu}(z, \bar{z})$ and $\bar{\partial} X^{\mu}(z, \bar{z})$.

Closed string solution:

$$X_{\mu}^{\tau} = \frac{1}{2} x^{\mu} + \frac{1}{2} l_s p^{\mu} (\tau - \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-2in(\tau - \sigma)}$$

which sat.
 $\bar{z} = e^{2i(\tau + \sigma)}$
 $\frac{1}{2} \ln \bar{z} = \tau + \sigma$

$$X_{\mu}^{\sigma} = \frac{1}{2} x^{\mu} - \frac{1}{2} l_s p^{\mu} (\tau - \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-2in(\tau + \sigma)}$$

$$X_{\mu}^{\tau} = \frac{1}{2} x^{\mu} - \frac{i}{2} l_s p^{\mu} \ln \bar{z} + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} \bar{z}^{-n}$$

$$\partial X(z) = -\frac{i}{2} l_s \alpha_0 \frac{1}{z} - \frac{i}{2} l_s \sum_{n \neq 0} \alpha_n^{\mu} z^{-n-1} = -\frac{i}{2} l_s \sum_{n \neq 0} \alpha_n^{\mu} z^{-n-1}$$

$$\bar{\partial} X(\bar{z}) = -\frac{i}{2} l_s \sum_{n \neq 0} \bar{\alpha}_n^{\mu} \bar{z}^{-n-1}$$

For $f(z) = \sum_n a_n (z-c)^n \in \mathbb{C}$, $a_n = \frac{1}{2\pi i} \oint d\omega \frac{f(\omega)}{(\omega-c)^{n+1}}$

Thus,

$$f(z) = \partial X = \sum_n \left(-\frac{i l_s}{2} \alpha_n^{\mu} \right) z^{-n-1} \Rightarrow -\frac{i l_s}{2} \alpha_n = \frac{1}{2\pi i} \oint d\omega \partial X(\omega) \omega^n$$

Check: $f(z) = \sum_n \left(\frac{1}{2\pi i} \oint d\omega \frac{f(\omega)}{(\omega-c)^{n+1}} \right) (z-c)^n = \sum_n \left(\frac{1}{n!} \frac{d^n}{d\omega^n} f(\omega) \right) (z-c)^n = \sum_n \left(\frac{(z-c)^n}{n!} \right) \partial^n f(c)$

$$f(z) = \sum_n \frac{z^n}{n!} \partial^n f(c) \quad \omega = z = c \quad b = z = c \quad \text{TAYLOR EXP.}$$

$$\alpha_n = \frac{1}{\pi l_s} \oint d\omega \partial X(\omega) \omega^n \quad \neq \quad \bar{\alpha}_n = \frac{1}{\pi l_s} \oint d\bar{\omega} \bar{\partial} X(\bar{\omega}) \bar{\omega}^n$$

$$\begin{aligned}
[\alpha_m^h, \alpha_n^h] &= \frac{1}{\pi^2 \alpha'} \left[\oint dz z^m \partial X^h(z) \cdot \oint dw \omega^n \partial X^h(w) \right] \\
&= \frac{1}{\pi^2 \alpha'} \left[\underbrace{\oint dz z^m \oint dw \omega^n \partial X^h(z) \partial X^h(w)}_{|z| > |w|} - \underbrace{\oint dz z^m \oint dw \omega^n \partial X^h(w) \partial X^h(z)}_{|w| > |z|} \right] \\
&\quad \text{fix } z \quad \text{then} \quad \text{detour}
\end{aligned}$$


$$\begin{aligned}
&= \frac{1}{\pi^2 \alpha'} \oint dz z^m \oint dw \omega^n \partial X^h(z) \partial X^h(w) = \frac{1}{\pi^2 \alpha'} \oint dz z^m \oint dw \left(\frac{dw}{dz} \right) \eta^{\mu\nu} \text{Res} \left(\frac{\omega^n}{(w-z)^2}, z \right) \\
&\quad = \frac{-\alpha' \eta^{\mu\nu}}{2} \frac{1}{(z-w)^2} + \dots \\
&\quad \begin{cases} = 0 & m+n-1 \geq 0 \\ \neq 0 & m+n-1 < 0 \end{cases}
\end{aligned}$$

for $m+n=0$, $\eta^{\mu\nu} \text{Res}(z^{-1}) \delta_{m+n,0} = \eta^{\mu\nu} \delta_{m+n,0}$

for $m+n=-1$, $\eta^{\mu\nu} \text{Res}(z^{-2}) \delta_{m+n,-1} = 0$

similarly for $\forall m+n \leq -1$

2.2 Consider a conformal field $\Phi(z)$ of dimension h and a mode expansion $\Phi(z) = \sum_{n=-\infty}^{\infty} \frac{\tilde{\Phi}_n}{z^{n+h}}$. Evaluate the commutator $[L_m, \tilde{\Phi}_n]$.

$$\begin{aligned}
T(z) &= \sum_n L_n z^{-n-2} \rightarrow L_n = \frac{1}{2\pi i} \oint dz \frac{T(z)}{z^{n+1}} \quad \& \quad \Phi(z) = \sum_n \tilde{\Phi}_n z^{-n-h} \rightarrow \tilde{\Phi}_n = \frac{1}{2\pi i} \oint dw \frac{\Phi(w)}{w^{n+h+1}} \\
&= \frac{1}{2\pi i} \oint dz T(z) z^{n+1} \quad \quad \quad = \frac{1}{2\pi i} \oint dw \Phi(w) w^{n+h-1}
\end{aligned}$$

$$\begin{aligned}
[L_m, \tilde{\Phi}_n] &= \frac{1}{(2\pi i)^2} \left(\underbrace{\oint dz z^{m+1} \oint dw \omega^{n+h-1} T(z) \Phi(w)}_{|z| > |w|} - \underbrace{\oint dz z^{m+1} \oint dw \omega^{n+h-1} \Phi(w) T(z)}_{|w| > |z|} \right) \quad \text{primary?} \\
&= \frac{1}{(2\pi i)^2} \oint dz z^{m+1} \oint dw \omega^{n+h-1} T(z) \Phi(w) = \frac{1}{(2\pi i)^2} \oint dz z^{m+1} \oint dw \omega^{n+h-1} \left(\dots + h \frac{\Phi(w)}{(z-w)^2} + \frac{\partial \Phi(w)}{(z-w)} + \dots \right) \\
&= \frac{1}{(2\pi i)^2} \oint dz z^{m+1} \left\{ \text{Res} \left(h \frac{\omega^{n+h-1} \Phi(w)}{(w-z)^2} \right) + \text{Res} \left(\frac{\omega^{n+h-1} \partial \Phi(w)}{(w-z)} \right) \right\} \\
&= \frac{1}{2\pi i} \oint dz z^{m+1} \left\{ h(n+h-1) z^{n+h-2} \Phi(z) + h z^{n+h-1} \partial \Phi(z) + z^{n+h-1} \partial^2 \Phi(z) \right\} \\
&= \frac{1}{2\pi i} \oint dz h(n+h-1) z^{m+n+h-1} \Phi(z) + \underbrace{(h+1) z^{m+n+h} \partial \Phi(z)}_{= (h+1)(m+n+h) z^{m+n+h-1} \Phi(z)} \\
&= h(n+h-1) \tilde{\Phi}_{m+n} - (h+1)(m+n+h) \tilde{\Phi}_{m+n} \\
&= \cancel{(h+1)} h - h m - \cancel{h m} - \cancel{h n} - m - n - h \tilde{\Phi}_{m+n} \\
&= (-2h - (h+1)m - n) \tilde{\Phi}_{m+n} \quad \text{messed up the order of } w \& z, \text{ since we have } T(z) \Phi(w) \text{ we want to do the } \oint dz \text{ integral first!} \\
&\quad \hookrightarrow \text{correct answer } (m(h-1) - n) \tilde{\Phi}_{m+n}
\end{aligned}$$

2.8 Let $\tilde{\Phi}(z)$ be a holomorphic primary field of conformal dimension h , as given above. The conformal vacuum satisfies $\tilde{\Phi}_n|0\rangle = 0$ for $n > -h$. Prove that $|\tilde{\Phi}\rangle = \tilde{\Phi}_{-h}|0\rangle$ is a highest-weight state.

ASIDE: TQFT 4.8.3

Suppose $\exists |\psi\rangle \neq 0$. $L_0|\psi\rangle = h|\psi\rangle$ & $\tilde{L}_0|\psi\rangle = \tilde{h}|\psi\rangle$, then $L_0 L_n |\psi\rangle = [L_0, L_n] |\psi\rangle + L_n L_0 |\psi\rangle = (h-n)L_n |\psi\rangle$
 $(0-n)L_n + 0$ $\hookrightarrow \begin{cases} n < 0: \text{raise energy} \\ n > 0: \text{lowers energy} \end{cases}$

For the spectrum to be bound below, need a primary state: $L_0|\psi\rangle = \tilde{L}_0|\psi\rangle = 0 \quad \forall n > 0$.
highest-weight state

$$L_n |\tilde{\Phi}\rangle = L_n \tilde{\Phi}_{-h} |0\rangle = [L_n, \tilde{\Phi}_{-h}] |0\rangle + \tilde{\Phi}_{-h} L_n |0\rangle$$

$$= (n(h-1) + h) \tilde{\Phi}_{-h} |0\rangle + \tilde{\Phi}_{-h} L_n |0\rangle$$

$\tilde{\Phi}_{-h} |0\rangle = 0$ $L_n |0\rangle = 0 \quad \forall n \geq -1$

$\forall n > -h \Rightarrow \forall n > 0$

$$= \begin{cases} 0 & n=0 \\ \text{---} & n > 0 \end{cases}$$

$$\tilde{L}_n |\tilde{\Phi}\rangle = \tilde{\Phi}_{-h} \tilde{L}_n |0\rangle = 0 \quad n > 0$$

$= 0 \quad n=0$

2.9 (i) Calculate the 2pt. functions $\langle 0 | \phi_i(z, \bar{z}) \phi_j(w, \bar{w}) | 0 \rangle$ for an arbitrary pair of primary fields w/ conformal weights $(h_i, \bar{h}_i)$ taking into account that the Virasoro generators L_0 & $\tilde{L}_0$ the in and out vacua $|0\rangle$ & $\langle 0|$.

Since $L_n |0\rangle = 0 \quad \forall n \geq -1$, we have that $\langle 0 | [L_n, \mathcal{O}] | 0 \rangle = 0$ for $n = -1, 0, 1$

Consider first 

$$[L_n, \phi(z)] = \frac{1}{2\pi i} \oint d\omega \omega^m \left(\underbrace{T(\omega) \phi(z)}_{|m| > |z|} - \underbrace{\phi(z) T(\omega)}_{|z| > |\omega|} \right) = \frac{1}{2\pi i} \oint d\omega \omega^m T(\omega) \phi(z) = \text{Res} \left(\eta \frac{\omega^m}{(z-\omega)^2} \phi(z) \right) + \text{Res} \left(\frac{\omega^m}{(z-\omega)} \partial \phi(z) \right)$$

$$= h \frac{\phi(z)}{(z-\omega)^2} + \frac{\partial \phi(z)}{(z-\omega)} + \dots$$

$$= h(n+1) z^n \phi(z) + z^{n+1} \partial \phi(z)$$

Let $G_{ij}(z, \bar{z}, w, \bar{w}) = \langle 0 | \phi_i(z, \bar{z}) \phi_j(w, \bar{w}) | 0 \rangle$.

Then, for L_{-1} ,

$$\langle 0 | [L_{-1}, \phi_i(z) \phi_j(w)] | 0 \rangle = \langle 0 | [L_{-1}, \phi_i(z)] \phi_j(w) | 0 \rangle + \langle 0 | \phi_i(z) [L_{-1}, \phi_j(w)] | 0 \rangle = (\partial_z + \partial_w) G(z, w) = 0$$

$= \partial_z \phi_i(z)$ $\partial_{\bar{z}} \phi_j(w)$

$$\Rightarrow G(z, w) = G(z, w) \quad \text{let } u = z-w \text{ & } v = z+w$$

$$\partial_z = \frac{\partial u}{\partial z} \partial_u + \frac{\partial v}{\partial z} \partial_v = \partial_u + \partial_v$$

$$\partial_w = \frac{\partial u}{\partial w} \partial_u + \frac{\partial v}{\partial w} \partial_v = -\partial_u + \partial_v$$

$(\partial_z + \partial_w) G(z, w) = 2\partial_v G(z, w) = 0 \Rightarrow G = G(v)$
 $G = G(w)$
 $= G(z-w)$

For L_0 ,

$$\langle 0 | [L_0, \phi_i(z) \phi_j(w)] | 0 \rangle = \langle 0 | \phi_i(z) [L_0, \phi_j(w)] | 0 \rangle$$

$$= h_i \phi_i(z) + z \partial \phi_i(z) \quad h_j \phi_j(w) + w \partial \phi_j(w)$$

$$= (h_i + h_j + z \partial_z + w \partial_w) G(z, w) = (h_i + h_j + z \partial_u + w \partial_v) G(u) = (h_i + h_j + u \partial_u) G(u) = 0$$

$$\Rightarrow u \frac{dG(u)}{du} = -(h_i + h_j) G(u) \rightarrow \int \frac{dG}{G} = -(h_i + h_j) \int \frac{du}{u} \rightarrow \ln(G) = -(h_i + h_j) \ln(u) + C' = \ln(C u^{-(h_i + h_j)})$$

$$\Rightarrow G_{ij}(z, w) = C_{ij} (z-w)^{-h_i - h_j}$$

For L_n ,

$$2n z \phi_j(z) + z^2 \partial \phi_j(z)$$

$$\langle \phi | [L_n, \phi_j(z)] \phi_j(w) | 0 \rangle + \langle \phi | \phi_j(z) [L_n, \phi_j(w)] | 0 \rangle$$

$$= (2n z + 2n j w) G_{ij}(z-w) + (z^2 \partial_z + w^2 \partial_w) G_{ij}(z-w)$$

$$\partial_z (G_{ij}(z-w)^{-h_i-h_j}) = -(h_i+h_j) \frac{G_{ij}}{(z-w)}$$

$$= 2(h_i z + h_j w) G_{ij} + (h_i + h_j) (-z^2 + w^2) \frac{G_{ij}}{z-w} = 0$$

$$\partial_w (\text{---}) = + (h_i + h_j) \frac{G_{ij}}{(z-w)}$$

$$\Rightarrow 2(h_i z + h_j w) - (h_i + h_j)(z+w) = h_i z + h_j w - h_i w - h_j z = (h_i - h_j) z - (h_i - h_j) w = (h_i - h_j)(z-w) \stackrel{0}{=} 0$$

$$\Rightarrow h_i = h_j$$

$\Rightarrow$ If ϕ_j is an orthonormal basis.

Therefore,
$$G_{ij}(z-w, z-w) = \frac{G_{ij}}{(z-w)^{2h} (z-w)^{2h}}$$

(3.10) (i) Show that in a unitary CFT, i.e. with positive definite Hilbert space, the central charge satisfies $c > 0$, and h of primary fields satisfies $h \geq 0$.

Hermitian Hamiltonian $\Rightarrow$ unitarity

$$T_{00} = \partial_0 X \cdot \partial_0 X - \frac{1}{2} \partial_0 \phi \cdot \partial_0 \phi$$

$$H = T_{00} = (\partial_0 X)^2 + \frac{1}{2} (-(\partial_0 \phi)^2 + (\partial_0 \psi)^2) = \frac{1}{2} (\partial_0 X)^2 + \frac{1}{2} (\partial_0 \psi)^2 \quad \text{---} = T_{00} + T_{0\bar{0}} = \sum_n L_n e^{-in\tau} + \bar{L}_n e^{-in\tau}$$

$$H^\dagger = H \Rightarrow L_n^\dagger = L_{-n}$$

Consider

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} n(n^2-1) \delta_{m+n,0}$$

$$\langle \phi | [L_m, L_n] | \phi \rangle = \langle \phi | (2n L_0 + \frac{c}{12} n(n^2-1)) | \phi \rangle = (2nh + \frac{c}{12} n(n^2-1)) \langle \phi | \phi \rangle$$

$$\langle \phi | L_{m-n} | \phi \rangle = |L_n \phi|^2 \geq 0$$

$$\hookrightarrow \text{for } n=1, \quad 2h \frac{\langle \phi | \phi \rangle}{20} \geq 0 \Rightarrow h \geq 0$$

$$\langle \phi | L_{-n} L_n | 0 \rangle = \langle \phi | [L_n, L_{-n}] | 0 \rangle = \frac{c}{12} n(n^2-1) \langle 0 | 0 \rangle \Rightarrow c \geq 0 \quad \text{(any non-trivial CFT has } c > 0)$$

(3.10) (ii) Show that $h = \bar{h} = 0$ if and only if $|\phi\rangle = |0\rangle$.

$$h = \bar{h} = 0 \Rightarrow |\phi\rangle = |0\rangle, \quad L_0 |\phi\rangle = h |\phi\rangle = 0, \quad \bar{L}_0 |\phi\rangle = \bar{h} |\phi\rangle = 0$$

$$L_n |\phi\rangle = 0 \quad n > 0$$

need to check $n=-1$,

$$\langle \phi | L_{-1} L_{-1} | \phi \rangle = \langle \phi | [L_{-1}, L_{-1}] | \phi \rangle = 2h \langle \phi | \phi \rangle = 0 \Rightarrow L_{-1} |\phi\rangle = 0 \quad \checkmark$$

$$|\phi\rangle = |0\rangle \Rightarrow h = \bar{h} = 0, \quad L_0 |\phi\rangle = L_0 |0\rangle = 0 \Rightarrow h = 0$$

$$\bar{L}_0 |\phi\rangle = \bar{L}_0 |0\rangle = 0 \Rightarrow \bar{h} = 0 \quad \checkmark$$

3.4) Verify the expression $\epsilon(2,\lambda) = -22(6\lambda^2 - 6\lambda + 1)$ by computing the CPE of T_{bc} with itself.

ASIDE: TONG 5.

2 interpretations of conformal symmetry: fixed background, global symmetry (Statistical Physics) $\neq$ dynamic background, local symmetry (String Theory)

Classically: $T_{ab}^{\text{cl}} = 0$ but in the quantum theory $\langle T_{ab}^{\text{cl}} \rangle = -\frac{c}{12} R$

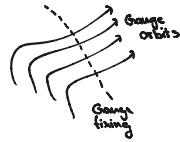
$\hookrightarrow$ different values for backgrounds related by a Weyl transformation.

$\rightarrow$ Need $c > 0$, even though we saw that $c > 0$ for non-trivial CFT.

5.1 PATH INTEGRAL

$$Z = \frac{1}{\text{Vol}} \int \mathcal{D}g \mathcal{D}X e^{-S_{\text{poly}}[X,g]} \xrightarrow{\text{Path integral}} Z = \frac{1}{\text{Vol}} \int \mathcal{D}g \mathcal{D}X e^{-S_{\text{poly}}[X,g]}$$

$\hookrightarrow$ physically distinct configs, divide by vol of gauge action



FADDEEV-POPOV METHOD

The gauge sym. diffeomorphisms & Weyl $g_{\mu\nu}(\sigma) \rightarrow g_{\mu\nu}(\sigma') = e^{2\omega(\sigma)} \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} g_{\mu\nu}(\sigma)$

Spec. we integrate over the gauge orbit of g , then at some point $\hat{g}^3 = g$

$$\int \mathcal{D}S \delta(g - \hat{g}^3) = \delta_{\text{FP}}[g] \longleftrightarrow \int dx \delta(f(x)) = \frac{1}{|f'|}$$

Weyl anomaly means that our original theory fails to be gauge invariant, we will show that Δ_{FP} also fails to be gauge invariant, BUT manages to cancel the original failure in certain circumstances.

$$Z[\hat{g}] = \frac{1}{\text{Vol}} \int \mathcal{D}g \mathcal{D}X e^{-S_{\text{poly}}[X,g]} \mathbb{1} = \frac{1}{\text{Vol}} \int \mathcal{D}S \mathcal{D}X \mathcal{D}g \Delta_{\text{FP}}[g] \delta(g - \hat{g}^3) e^{-S_{\text{poly}}[X,g]} = \frac{1}{\text{Vol}} \int \mathcal{D}S \mathcal{D}X \Delta_{\text{FP}}[\hat{g}^3] e^{-S_{\text{poly}}[X,\hat{g}^3]}$$

$\downarrow$
invariant under diffeomorphisms $\checkmark$
 S invariant under Weyl $\checkmark$ $\mathcal{D}X, \Delta_{\text{FP}}$ not invariant under Weyl $\times$

If everything conspires such that it is invariant under both diffe. & Weyl, then

$$Z[\hat{g}] = \frac{1}{\text{Vol}} \int \mathcal{D}S \mathcal{D}X \Delta_{\text{FP}}[\hat{g}] e^{-S_{\text{poly}}[X,\hat{g}]} = \int \mathcal{D}X \Delta_{\text{FP}}[\hat{g}] e^{-S_{\text{poly}}[X,\hat{g}]}$$



$$Z[\hat{g}] = \int \mathcal{D}X \mathcal{D}b \mathcal{D}c e^{-S_{\text{poly}}[X,\hat{g}] - S_{\text{ghost}}[b,c,\hat{g}]}$$

$\hookrightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d\sigma \sqrt{g} b_{\mu\nu} \partial^\mu c^\nu$

$\rightarrow$ Ghosts cancel unphysical gauge dof., leaving only the D-2 transverse modes of X^μ . Unlike lightcone quantisation, they achieve this in a way that preserves Lorentz invariance.

In the conformal gauge, $\hat{g}_{\mu\nu} = e^{2\omega} \delta_{\mu\nu}$, the ghost action simplifies even further

$$\hat{g} = e^{2\omega}$$

$$z = \sigma' + i\sigma'' \quad \bar{z} = \sigma' - i\sigma''$$

$$\Rightarrow \sigma' = \frac{1}{2}(z + \bar{z}) \quad \sigma'' = \frac{i}{2}(z - \bar{z})$$

$$\Rightarrow d\sigma' = \frac{1}{2}(dz + d\bar{z}) \quad d\sigma'' = \frac{i}{2}(dz - d\bar{z}) \quad d\sigma = d\sigma' \wedge d\sigma'' = \frac{-i}{4} (dz \wedge dz - dz \wedge d\bar{z} + d\bar{z} \wedge dz - d\bar{z} \wedge d\bar{z})$$

$$= \frac{i}{2} \frac{dz \wedge d\bar{z}}{\in \mathbb{R}} = \frac{1}{2} (i dz \wedge d\bar{z}) = \frac{1}{2} \frac{d^2 z}{\in \mathbb{R}}$$

$$b_{\mu\nu} \nabla^\mu c^\nu = b_{23} \nabla^2 c^3 + b_{32} \nabla^3 c^2 + b_{33} \nabla^3 c^3 + b_{33} \nabla^3 c^3$$

$$\nabla^2 = g^{\mu\nu} \nabla_\mu = \frac{1}{g} \left((d\sigma')^2 + (d\sigma'')^2 \right) = \frac{1}{4} (dz^2 + 2dzd\bar{z} + d\bar{z}^2) = \frac{1}{4} (b_{22} + b_{33}) = 0$$

$$= 2e^{2u} \nabla_{\bar{z}}$$

$$-\frac{1}{4} (dz^2 - 2dzd\bar{z} + d\bar{z}^2) = d\bar{z}d\bar{z} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow g_{\mu\nu} = \frac{1}{2} e^{2u} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g^{\mu\nu} = 2e^{-2u} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \text{ghost action: } S_{\text{ghost}} = \frac{1}{2\pi} \int d\sigma \sqrt{g} b_{\mu\nu} \nabla^\mu c^\nu = \frac{1}{2\pi} \int d^2 z \frac{1}{2} e^{2u} (2e^{-2u}) (b_{22} \nabla_{\bar{z}} c^2 + b_{33} \nabla_z c^3)$$

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

$$\text{Now consider: } \nabla_{\bar{z}} c^2 = \partial_{\bar{z}} c^2 + \Gamma_{\bar{z}\alpha}^{\beta} c^\alpha \quad \alpha = 2, 3$$

$$\Gamma_{\bar{z}\alpha}^{\beta} = \frac{1}{2} g^{\beta\gamma} (\partial_{\bar{z}} g_{\gamma\alpha} + \partial_\alpha g_{\gamma\bar{z}} - \partial_\gamma g_{\alpha\bar{z}})$$

$$= \frac{1}{2} g^{\beta\gamma} (\partial_{\bar{z}} g_{\gamma\alpha} - \partial_{\bar{z}} g_{\gamma\alpha}) = 0 \quad \alpha = 2, 3$$

$$\Rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d^2 z \left(b_{22} \frac{1}{2} e^{2u} \partial_{\bar{z}} c^2 + b_{33} \frac{1}{2} e^{2u} \partial_z c^3 \right) \neq S[\omega] \rightarrow \text{Weyl invariant.}$$

$$\text{eom. } \partial\bar{\partial}b = \partial\bar{\partial}c = \partial\bar{\partial}\bar{c} = 0$$

$$\rightarrow \text{The energy-momentum tensor of } S_{\text{ghost}} \text{ is: } T(\bar{z}) = 2(\partial\bar{\partial}b)b + c(\partial\bar{\partial}b) \quad \bar{T}(z) = 2(\bar{\partial}z\bar{c})\bar{b} + \bar{c}(\bar{\partial}z\bar{b})$$

We find the OPE in the usual way...

$$0 = \int D\bar{b} Dc \frac{\delta}{\delta b(\sigma)} \left(e^{-\frac{1}{2\pi} \int d^2\omega b(\omega) \bar{c}(\omega)} \right) = \int D\bar{b} Dc e^{-\frac{1}{2\pi} \int d^2\omega b(\omega) \bar{c}(\omega)} \left(-\frac{\delta S_{\text{ghost}}}{\delta b(\sigma)} b(\sigma) + \bar{c}(\sigma - \sigma') \right) \Rightarrow \bar{\partial}c(\sigma) b(\sigma) = 2\pi \bar{c}(\sigma - \sigma')$$

$$\frac{1}{2\pi} \bar{\partial}c \rightarrow -(\bar{\partial}b)c \rightarrow c(\bar{\partial}b)$$

$$0 = \int D\bar{b} Dc \frac{\delta}{\delta c(\sigma)} \left(e^{-\frac{1}{2\pi} \int d^2\omega b(\omega) \bar{c}(\omega)} \right) = \int D\bar{b} Dc e^{-\frac{1}{2\pi} \int d^2\omega b(\omega) \bar{c}(\omega)} \left(-\frac{1}{2\pi} (\bar{\partial}b) c(\sigma) + \bar{b}(\sigma - \sigma') \right) \Rightarrow \bar{\partial}b(\sigma) c(\sigma) = 2\pi \bar{b}(\sigma - \sigma')$$

$$\text{Using } \bar{\partial}(\frac{1}{z}) = 2\pi \delta(z, \bar{z}) \xrightarrow{\text{"proof."}}$$

$$f(\sigma, \bar{\sigma}) = \int d^2 z f(z, \bar{z}) \delta(z, \bar{z}) \quad \text{NORMALLY HANDLED BY}$$

$$\text{Consider: } \int d^2 z f(z, \bar{z}) \bar{\partial}(\frac{1}{z}) = - \int d^2 z \bar{\partial}(f(z, \bar{z})) \frac{1}{z} = - \int d^2 z \bar{\partial}(f(z, \bar{z})) \frac{1}{z} = - \int d^2 z \frac{\bar{\partial}(f(z, \bar{z}))}{z} \sim f(\sigma, \bar{\sigma})$$

$$\bar{\partial}_z (c(z) b(\omega)) = 2\pi \delta(z - \omega) \Rightarrow c(z) b(\omega) = \frac{1}{z - \omega} + \dots$$

$$b(\omega) c(z) = \frac{1}{\omega - z} + \dots$$

$$T \text{ for ghost in } q\text{-theory: } T(z) = 2:(\partial c)(\bar{\partial} b)(z): + :c(\bar{\partial})(\partial b)(z):$$

$$T(z)T(\omega) = 4: \partial c(\bar{\partial} b)(z): : \partial c(\bar{\partial} b)(\omega): + 2: \partial c(\bar{\partial} b)(z): : c(\bar{\partial})(\partial b)(\omega): + 2: c(\bar{\partial})(\partial b)(z): : \partial c(\bar{\partial} b)(\omega): + :c(\bar{\partial})(\partial b)(z): : c(\bar{\partial})(\partial b)(\omega):$$

$$= 4: \langle \partial c(\bar{\partial} b)(z) : \langle \bar{\partial} b(\bar{\partial} c)(\omega) \rangle + 4: \partial c(\bar{\partial} b)(z) : \langle c(\bar{\partial})(\partial b)(\omega) \rangle + 4: b(\bar{\partial} c)(z) : \langle \partial c(\bar{\partial} b)(\omega) \rangle$$

$$\partial_z \langle c(\bar{\partial} b)(\omega) \rangle = \frac{-1}{(z - \omega)^2} \quad \partial_z \langle b(\bar{\partial} c)(\omega) \rangle = \frac{+1}{(z - \omega)^2}$$

$$\partial_z \partial_\omega \langle c(z) b(\omega) \rangle = \partial_z \partial_\omega \left(\frac{1}{z - \omega} \right) = \partial_z \left(\frac{-1}{(z - \omega)^2} \right) = \partial_z \left(\frac{1}{(z - \omega)^2} \right) = \frac{-2}{(z - \omega)^3}$$

$$= \frac{-4}{(z-w)^4} + \frac{+4: \partial \bar{c} b a \eta :}{(z-w)^4} - \frac{4: b \bar{c} a \eta c \eta :}{(z-w)^4} + 2 \left(\frac{-2}{(z-w)^3} \right) \left(\frac{1}{z-w} \right) + \frac{-4: b \bar{c} a \eta c \eta :}{(z-w)^4} + \frac{2: \partial \bar{c} b a \eta c \eta :}{z-w}$$

$$+ 2 \left(\frac{1}{(z-w)^3} \right) \left(\frac{-2}{(z-w)^2} \right) + \frac{-4: c \bar{b} a \eta :}{(z-w)^4} + \frac{2: \partial \bar{c} b a \eta :}{z-w} + \left(\frac{1}{(z-w)^2} \right) \left(\frac{-1}{(z-w)^2} \right) + \frac{-2: c \bar{b} a \eta c \eta :}{(z-w)^4} + \frac{2: \partial \bar{c} b a \eta c \eta :}{(z-w)^4} + \dots$$

$$f(z+w) = f(w) + a \partial f(w) + \frac{1}{2!} a^2 \partial^2 f(w) + \frac{1}{3!} a^3 \partial^3 f(w), \quad a = z-w$$

$$f(z) = f(w) + (z-w) \partial f(w) + \frac{1}{2!} (z-w)^2 \partial^2 f(w) + \dots$$

$$= \frac{-13}{(z-w)^4} + \frac{4: \partial \bar{c} a \eta b \eta :}{(z-w)^4} + \frac{4: \partial \bar{c} a \eta b \eta :}{z-w} - 4 \frac{: b \bar{c} c :}{(z-w)^4} - 4 \frac{: \partial \bar{b} c :}{(z-w)^4}$$

$$- 4 \frac{: b \bar{c} c :}{(z-w)^4} - 4 \frac{: \partial \bar{b} c :}{(z-w)^4} - 2 \frac{: \partial^2 \bar{b} c :}{z-w} + 2 \frac{: \partial c \partial \bar{b} :}{z-w}$$

$$- 4 \frac{: c \bar{b} c :}{(z-w)^4} - 4 \frac{: (b \bar{c}) b :}{(z-w)^4} - 2 \frac{: (\partial \bar{c}) b :}{(z-w)^4} + 2 \frac{: \partial \bar{b} \partial c :}{z-w} - \frac{: c \bar{b} c :}{(z-w)^4} - \frac{: \partial c \partial \bar{b} :}{z-w} + \frac{: \partial \bar{b} c :}{(z-w)^4} + \frac{: \partial^2 \bar{b} c :}{(z-w)^4} + \dots$$

$$= \frac{-13}{(z-w)^4} + \frac{1}{(z-w)^4} \left(-4: c \bar{b} c : - 4: b \bar{c} c : \right) + \frac{1}{(z-w)^4} \left(4: \partial \bar{c} b : - 4: b \bar{c} c : - 4: (\partial \bar{b} c) : - 4: (\partial c \bar{b}) : - 4: (\partial \bar{b} c) : + (\partial \bar{b} c) : \right)$$

$$+ \frac{1}{(z-w)^4} \left(4: \partial \bar{c} b : - 4: (\partial \bar{b} c) : - 2: (\partial^2 \bar{b} c) : + 2: (\partial c \partial \bar{b}) : - 2: (\partial \bar{c} b) : + 2: (\partial \bar{b} \partial c) : - (\partial c \partial \bar{b}) : + (\partial^2 \bar{b} c) : \right)$$

$$= \frac{-13}{(z-w)^4} + \frac{1}{(z-w)^4} (4: \partial \bar{c} b : + 2: c \partial \bar{b} :) + \frac{1}{(z-w)^4} (2: \partial \bar{c} b : + 3: \partial c \partial \bar{b} : + : c \partial \bar{b} c :) + \dots$$

$$= \frac{-13}{(z-w)^4} + \frac{2: T \eta :}{(z-w)^4} + \frac{\partial T \eta}{z-w} + \dots \sim T \bar{c} T \eta = \frac{c \bar{c}}{(z-w)^4} + \frac{2: T \eta :}{(z-w)^2} + \frac{\partial T \eta}{z-w} + \dots \Rightarrow c = -2b$$

ASIDE:

To achieve c.c.o. we simply need some CPT with $c=2b$, for example 26 free scalar fields.

3.12 Verify the property $Q_B^2=0$ of the BRST charge by anticommuting two of the integral representations and using the various OPEs.

ASIDE:

$$\text{from } \partial \bar{c} \bar{c} = 0 \Rightarrow \partial \bar{c} \bar{c} = \partial \bar{c} \bar{c} = 0$$

$$\partial \bar{c} = \partial \bar{c} = 0$$

$$-\partial \bar{b} c = -\partial \bar{b} c$$

no anomaly because CPE of T w/ itself has no central charge, or rather $c_A + c_{bc} = 26 - 26 = 0$.

The full BRST action is: $S_B = \frac{1}{2\pi} \int d^2 z \left(2\partial \bar{c} \partial X + b \bar{\partial} c + \bar{b} \partial c \right) \rightarrow \text{energy-momentum tensor } T(z) = T_X(z) + T_{bc}(z)$

Remaining BRST symmetry: $\delta X^\mu(z) = \eta c \partial X^\mu \quad \delta \bar{c}(z) = \eta \bar{c} \partial \bar{c}$

$$\delta b(z) = \eta T = \eta (T_X + T_b) \quad \delta \bar{b}(z) = \eta \bar{T}$$

$$\delta c(z) = \eta c \partial c \quad \delta \bar{c}(z) = \eta \bar{c} \partial \bar{c}$$

Check:

$$\delta S_B = \int d^2 z \left(2\partial(\delta \bar{c}) \cdot \partial X + 2\partial \bar{c} \cdot \partial(\delta X) + (\delta b) \bar{\partial} c + b \bar{\partial}(\delta c) + (\delta \bar{b}) \partial c + \bar{b} \partial(\delta c) \right)$$

$$= \int d^2 z \left(2\partial(\eta \bar{c} \partial \bar{c}) \cdot \partial X + 2\partial \bar{c} \cdot \partial(\eta c \partial X + \eta \bar{c} \partial \bar{X}) + (\eta T) \bar{\partial} c + b \bar{\partial}(\eta c \partial c) + (\eta \bar{T}) \partial c + \bar{b} \partial(\eta \bar{c} \partial \bar{c}) \right)$$

$$= \int d^2 z \left(2\eta \partial c \partial X \cdot \partial \bar{X} + 2\eta c \partial^2 X \cdot \partial \bar{X} + 2\eta \partial \bar{c} \partial X \cdot \partial \bar{X} + 2\eta \bar{c} \partial^2 \bar{X} \cdot \partial \bar{X} + \eta T \bar{\partial} c + \eta \bar{T} \partial c + \eta b \bar{\partial} c \partial c + \eta b c \partial \bar{\partial} c \right.$$

$$\left. + 2\eta \bar{\partial} c \partial X \partial X + 2\eta c \partial X \cdot \partial \partial X + 2\eta \bar{\partial} \bar{c} \partial X \cdot \partial \bar{X} + 2\eta \bar{c} \partial X \cdot \partial \partial \bar{X} + \eta \bar{T} \partial c + \eta \bar{T} \partial c + \eta \bar{b} \partial \bar{c} \partial \bar{c} + \eta \bar{b} \bar{c} \partial \bar{c} \right)$$

$$= \int d^2z \left(2\eta \partial (c \partial x \bar{\partial} x) + 2\eta \bar{\partial} (\bar{c} \bar{\partial} x \partial x) + 2\eta \left(\frac{\partial \bar{c} \bar{\partial} x \bar{\partial} x}{-\bar{\partial} x} + \frac{\bar{\partial} c \partial x \partial x}{-\partial x} \right) + \eta \left(\bar{T}_x \partial \bar{c} + T_x \bar{\partial} c \right) + \eta \left(-2b \partial c \bar{\partial} c + c \partial b \bar{\partial} c \right) - 2\bar{b} \bar{\partial} c \partial c + c \bar{\partial} b \partial c \right)$$

$$= \int d^2z \left(\eta \left(\underbrace{-2b \partial c \bar{\partial} c - \partial b c \bar{\partial} c - b \bar{\partial} c \partial c + bc \bar{\partial} c}_{=-\partial(bc\bar{\partial}c)} - \underbrace{2\bar{b} \bar{\partial} c \partial c - \bar{\partial} b c \partial c + \bar{b} \partial c \bar{\partial} c + \bar{b} \bar{\partial} c \partial c}_{=-\bar{\partial}(\bar{b}c\partial c)} \right) \right)$$

$$= 0$$

By Noether's thm. we have a conserved current

$$S = \int d^2z \left(2\partial x \bar{\partial} \eta + b \bar{\partial} c + \bar{b} \partial c \right)$$

$$\delta x = -\partial(bc\bar{\partial}c) - \bar{\partial}(\bar{b}c\partial c)$$

$$j(z) = \frac{\partial \bar{x}}{\partial \partial x} \delta x + \frac{\partial \bar{x}}{\partial \bar{\partial} x} \delta x + \frac{\partial \bar{x}}{\partial \partial c} \delta c + \frac{\partial \bar{x}}{\partial \bar{\partial} c} \delta c + \frac{\partial \bar{x}}{\partial \partial b} \delta b + \frac{\partial \bar{x}}{\partial \bar{\partial} b} \delta b + bc \bar{\partial} c$$

on shell?

$$= 2\bar{\partial} x \eta c \partial x + 2\partial x \eta c \bar{\partial} x + 0 + b \eta c \partial c + \eta b c \bar{\partial} c$$

$$= 2\eta \partial x \partial x - \eta b c \partial c - c \bar{T}_x + b c \partial c \quad T = -i\partial^2$$

$$\delta S = 0 \Rightarrow \delta x = \partial \eta \bar{T}$$

$$\delta x = \frac{\partial \bar{x}}{\partial \phi} \delta \phi + \frac{\partial \bar{x}}{\partial \partial \phi} \delta \partial \phi = \frac{\partial \bar{x}}{\partial \phi} \delta \phi + \partial \left(\frac{\partial \bar{x}}{\partial \partial \phi} \delta \phi \right) - \partial \left(\frac{\partial \bar{x}}{\partial \partial \phi} \right) \delta \phi$$

$$\Rightarrow \partial \left(\frac{\partial \bar{x}}{\partial \partial \phi} \delta \phi - T \right) = 0$$

Consider $\{Q, Q\} = 2Q^2$

$$\{Q, Q\} = \frac{1}{(2\pi)^2} \oint dz \oint dw j(z) \oint dw j(w) + \frac{1}{(2\pi)^2} \oint dw \oint dz j(w) \oint dz j(z) \quad w/ \quad j(z) = c T_x + b c \partial c$$

$$= \frac{1}{(2\pi)^2} \oint dz \oint dw j(z) j(w) - \oint dw \oint dz j(z) j(w)$$

$$= \frac{1}{(2\pi)^2} \oint dw \oint dz j(z) j(w)$$

$$j(z) j(w) = -j(w) j(z)$$



$$j(z) j(w) = \left(c(w) T_x(z) + b(w) c(w) \partial c(w) \right) \left(c(z) T_x(w) + b(z) c(z) \partial c(z) \right)$$

= ... long ...

Currently missing (3.12) - (3.16).

R. Bouchard

22/10/2025