

$$= \int dt \partial_t (-e^2 \tilde{z} \dot{x}^I \dot{x}_I) + 2i \partial_t (e^2 \tilde{z} \dot{x}^I \dot{\psi}_I \chi) + 0 = 0 //$$

(ii) Show explicitly that $\tilde{S}_0$ is invariant under local susy transformations

$$\delta x^I = i \epsilon \psi^I \quad \delta \psi^I = \frac{1}{2\epsilon} (\dot{x}^I - i \chi \psi^I) \epsilon, \quad \delta \chi = \dot{\epsilon}, \quad \delta \epsilon = -i \chi \epsilon.$$

$$\tilde{S}_0 = \int dt \left(\frac{1}{2} \dot{e}^2 \dot{x}^I \dot{x}_I + i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi - i \dot{\psi}^I \dot{\psi}_I \right)$$

$$\begin{aligned} \delta \tilde{S}_0 &= \int dt \left(-\frac{1}{2} \dot{e}^2 (\delta e) \dot{x}^I \dot{x}_I + \dot{e}^2 \dot{x}^I (\delta \dot{x}_I) - i \dot{e}^2 (\delta e) \dot{x}^I \dot{\psi}_I \chi + i \dot{e}^2 (\delta \dot{x}^I) \dot{\psi}_I \chi + i \dot{e}^2 \dot{x}^I (\delta \dot{\psi}_I) \chi + i \dot{e}^2 \dot{x}^I \dot{\psi}_I (\delta \chi) + 2i \dot{\psi}^I \delta \dot{\psi}_I \right) \\ &= \int dt \left(\underbrace{-\frac{1}{2} \dot{e}^2 (-i \chi \epsilon) \dot{x}^I \dot{x}_I}_{-\frac{1}{2} \dot{e}^2 \dot{x}^I \dot{x}_I \chi \epsilon} + \underbrace{\dot{e}^2 \dot{x}_I (i \epsilon \psi^I + i \epsilon \dot{\psi}^I)}_{i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi \epsilon + i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi \epsilon} - i \dot{e}^2 (-i \chi \epsilon) \dot{x}^I \dot{\psi}_I \chi + i \dot{e}^2 (i \epsilon \dot{\psi}^I + i \epsilon \dot{\psi}^I) \dot{\psi}_I \chi + i \dot{e}^2 \dot{x}_I \frac{1}{2} \dot{e}^2 (\dot{x}^I - i \chi \psi^I) \epsilon \chi}_{\frac{1}{2} \dot{e}^2 \dot{x}^I \dot{x}_I \chi \epsilon} \right. \\ &\quad \left. + \underbrace{i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi \epsilon}_{i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi \epsilon} + \underbrace{i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi \epsilon}_{i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi \epsilon} + \underbrace{2i \dot{\psi}^I \delta \dot{\psi}_I}_{2i \dot{\psi}^I \dot{\psi}_I \chi \epsilon} \right) \\ &= 0 // \end{aligned}$$

(iii) Show that in the gauge $e=1, \chi=0$ one recovers S_0 w/ the constraints: $\dot{x}^2=0, \dot{x} \cdot \dot{\psi}=0$

$$\begin{aligned} \text{from } e: \quad & -\frac{1}{2} \dot{e}^2 \dot{x}^I \dot{x}_I - i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi = 0 \quad \& \quad \chi: \quad i \dot{e}^2 \dot{x}^I \dot{\psi}_I \chi = 0 \\ \Rightarrow \quad & \dot{x} \cdot \dot{x} = 0 \quad \quad \quad \Rightarrow \quad \dot{x} \cdot \dot{\psi} = 0 // \end{aligned}$$

$$\tilde{S}_0(e=1, \chi=0) = \int dt \frac{1}{2} \dot{x} \cdot \dot{x} - i \dot{\psi}^I \dot{\psi}_I = S_0 //$$

(4.3) Consider quantization of the superparticle action in Problem 4.1.

(i) Show that canonical quantization gives the equal τ (anti) commutation relations

$$[\dot{x}^I, \dot{x}^J] = i \eta^{IJ} \quad \& \quad \{\dot{\psi}^I, \dot{\psi}^J\} = \eta^{IJ}$$

$$S_0 = \int dt \frac{1}{2} \dot{x} \cdot \dot{x} - i \dot{\psi}^I \dot{\psi}_I \quad \longrightarrow \quad P^I = \frac{\partial \mathcal{L}}{\partial \dot{x}_I} = \dot{x}^I \quad \& \quad \Pi^I = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_I} = i \dot{\psi}^I$$

$$\begin{aligned} \text{By definition,} \quad & \{\dot{x}^I, P^J\} = \frac{\partial \dot{x}^I}{\partial \dot{x}^J} \frac{\partial P^J}{\partial P^I} - \frac{\partial P^J}{\partial \dot{x}^I} \frac{\partial \dot{x}^I}{\partial P^J} = \delta^I_J \eta^{IJ} = \eta^{IJ} \\ \Rightarrow \quad & \{\dot{x}^I, \dot{x}^J\} = \eta^{IJ} \quad \xrightarrow{\text{quantum}} \quad [\dot{x}^I, \dot{x}^J] = i \eta^{IJ} // \end{aligned}$$

$$\begin{aligned} \text{Similarly,} \quad & \{\dot{\psi}^I, \Pi^J\} = \eta^{IJ} \\ \Rightarrow \quad & \{\dot{\psi}^I, i \dot{\psi}^J\} = \eta^{IJ} \quad \xrightarrow{\text{quantum}} \quad \{\dot{\psi}^I, \dot{\psi}^J\} = i \eta^{IJ} \Leftrightarrow \{\dot{\psi}^I, \dot{\psi}^J\} = \eta^{IJ} // \end{aligned}$$

(ii) Explain why this describes a space-time fermion.

$\dot{x}^I(\tau)$ describe position dof. & $\dot{\psi}^I(\tau)$ describe spin dof.

$\rightarrow$ since $\{\dot{\psi}^I, \dot{\psi}^J\} = \eta^{IJ}$ we have a Clifford spin algebra $\rightarrow$ fermion. //

(iii) Explain the significance of the constraints $\dot{X}^2 = \dot{X} \cdot \dot{X} = 0$ in 4.2.

$\dot{X}^2 = \dot{X} \cdot \dot{X} = 0 \Rightarrow$ massless particle

$\dot{X} \cdot X = 0 \Rightarrow$ spin is transverse to momentum.

(49) Show the invariance of $S = -\frac{1}{2\pi} \int d\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^I \rho^\alpha \partial_\alpha \psi_I - 2B\partial\bar{\psi} \right)$ under $\delta X^\mu = \xi \psi^I$, $\delta \psi^I = \rho^\alpha \partial_\alpha X^\mu \xi + \partial^\mu \xi$, $\delta B = \xi \rho^\alpha \partial_\alpha \psi^I$

NOTES: $\rho^0 \cdot \rho^0 = -2$
 $\rho^0 \cdot \rho^1 = 0$

$\rho^\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\rho^\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\psi_\alpha^I = \begin{pmatrix} \psi_\alpha^1 \\ \psi_\alpha^2 \end{pmatrix}$, $\bar{\psi} = \psi^\dagger \rho^0$
 $S_\xi = -\frac{1}{2\pi} \int d\sigma \left[\bar{\psi}^I \rho^\alpha \partial_\alpha \psi_I + \frac{1}{2} \left(d\sigma^\mu \psi_\mu \cdot \partial_\mu \psi + \psi_\mu \cdot \partial_\mu \psi \right) \right]$ $\text{eqm: } \partial_\mu \psi_\mu = \partial_\mu \psi_\mu = 0$ Weyl condition

$\partial^\mu \eta_\mu = 2\eta^\alpha \partial_\alpha \eta_\alpha = \partial_\alpha \eta_\alpha - \partial_\alpha \eta_\alpha = 0$ $\hookrightarrow$ thus ψ_μ are Majorana-Weyl in 2d.

$V(\sigma, \theta^\alpha) = X(\sigma) + \theta^\alpha \eta_\alpha(\sigma) + \theta^\alpha \theta^\beta C(\sigma)$

$\delta \psi = (\theta^\alpha \rho^\alpha) i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = i(\theta^\alpha \rho^\alpha) \begin{pmatrix} \psi^2 \\ \psi^1 \end{pmatrix} = -i\theta^\alpha \psi^\alpha + i\theta^\beta \psi^\beta$
 $\rightarrow \begin{pmatrix} \delta \psi^1 \\ \delta \psi^2 \end{pmatrix} = \begin{pmatrix} -i\psi^2 \\ i\psi^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$ $\rightarrow \frac{1}{2} \delta \theta \delta = \frac{1}{2} (\theta^\alpha \rho^\alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \frac{1}{2} (-\theta^\alpha \theta^\alpha + \theta^\alpha \theta^\alpha) = i\theta^\alpha \theta^\beta$

$\delta S = -\frac{1}{2\pi} \int d\sigma \left(-2\partial_\alpha \theta^\alpha X \cdot \delta X + \delta \bar{\psi}^I \rho^\alpha \partial_\alpha \psi_I + \bar{\psi}^I \rho^\alpha \partial_\alpha \delta \psi_I - 2B\delta\bar{\psi} \right)$
 $= -\frac{1}{2\pi} \int d\sigma \left(-2\partial_\alpha \theta^\alpha X (\xi \psi) + \underbrace{\delta \bar{\psi}^I \rho^\alpha \partial_\alpha \psi_I}_{B \xi \rho^\alpha \partial_\alpha \psi} + \underbrace{\bar{\psi}^I \rho^\alpha \partial_\alpha \delta \psi_I}_{\bar{\psi}^I \rho^\alpha \partial_\alpha (B \xi)} - 2B \xi \rho^\alpha \partial_\alpha \psi \right)$
 $= -B(\partial_\alpha \bar{\psi}) \rho^\alpha \xi = B \xi \rho^\alpha \partial_\alpha \psi$

LHS: $\bar{\psi}^I \rho^\alpha \psi_I = -\bar{\psi}^I \rho^\alpha \psi_I$

For $\alpha=0$,

LHS = $i(\psi^1 \psi^2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = i(-\psi^1 \psi^1 + \psi^2 \psi^2)$ ✓

RHS = $i(\psi^1 \psi^2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = i(-\psi^1 \psi^1 + \psi^2 \psi^2)$

For $\alpha=1$,

LHS = $i(\psi^1 \psi^2 + \psi^2 \psi^1)$ ✓

RHS = $i(\psi^1 \psi^2 + \psi^2 \psi^1)$

Similarly,

$\delta S = -\frac{1}{2\pi} \int d\sigma \left(-2\partial^\alpha \theta_\alpha X (\delta X) + (\delta \bar{\psi}) \rho^\alpha \partial_\alpha \psi + \bar{\psi} \rho^\alpha \partial_\alpha (\delta \psi) - 2B\delta\bar{\psi} \right)$
 $= \frac{1}{2\pi} \int d\sigma \left(\partial^\alpha \theta_\alpha X (\xi \psi) + \underbrace{\delta \bar{\psi} \rho^\alpha \partial_\alpha \psi}_{-2\bar{\psi} \rho^\alpha \partial_\alpha \psi} + \underbrace{\bar{\psi} \rho^\alpha \partial_\alpha \delta \psi}_{-2\bar{\psi} \rho^\alpha \partial_\alpha \psi} - 2B\delta\bar{\psi} \right) = 0$
 $\text{if } \alpha=0, 2\rho^0 \rho^0 = 2\eta^0 \eta^0$ ✓
 $\alpha=1, \rho^1 \rho^1 = -\rho^0 \rho^0 \rightarrow \partial_\mu \partial_\mu X (\bar{\psi} \rho^1 \rho^1 \xi) = 0$

4.3 Derive the mass formulas for states in the R & NS sector of the RNS open superstring.

NOTES:

RNS string: $T_{\alpha\beta} = 2\alpha' \dot{x}_\mu \dot{x}_\nu \eta^{\mu\nu} + \frac{1}{2} \dot{\psi}^\dagger \rho_\beta \partial_\alpha \psi - (\text{trace})$ Energy-momentum (translation)

$J_\alpha^\alpha = -\frac{1}{2} (\rho^\beta \rho^\alpha \psi)_\alpha \partial_\beta x^\alpha$ Supercurrent (susy)

Bosonic EC are the same as before. For fermions,

$$S_F \sim \int d\sigma (\psi \partial_- \psi + \psi \partial_+ \psi)$$

$$\delta S_F = \int d\sigma (\psi \delta \psi - \psi \delta \psi)_{\sigma=\pi} - (\psi \delta \psi - \psi \delta \psi)_{\sigma=0}$$

Open string

Each end needs to vanish separately $\rightarrow \psi_+ = \pm \psi_-$ convention

Let $\psi_+^\dagger|_{\sigma=0} = \psi_-^\dagger|_{\sigma=0}$, then there are 2 cases for the other end:

- Ramond boundary condition: $\psi_+^\dagger|_{\sigma=\pi} = \psi_-^\dagger|_{\sigma=\pi}$ Gives rise to space-time fermions

$$\left. \begin{aligned} \psi_-^\dagger(\sigma, \tau) &= \frac{1}{\alpha'} \sum_{n \in \mathbb{Z}} d_n^\dagger e^{in(\tau-\sigma)} \in \mathbb{R} \\ \psi_+^\dagger(\sigma, \tau) &= \frac{1}{\alpha'} \sum_{n \in \mathbb{Z}} d_n^\dagger e^{in(\tau+\sigma)} \in \mathbb{R} \end{aligned} \right\} \text{Majorana condition} \rightarrow d_n^\dagger = d_n^\dagger$$

- Neveu-Schwartz boundary condition: $\psi_+^\dagger|_{\sigma=\pi} = -\psi_-^\dagger|_{\sigma=\pi}$ Gives rise to space-time bosons

$$\psi_-^\dagger(\sigma, \tau) = \frac{1}{\alpha'} \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n^\dagger e^{-in(\tau-\sigma)}$$

$$\psi_+^\dagger(\sigma, \tau) = \frac{1}{\alpha'} \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n^\dagger e^{-in(\tau+\sigma)}$$

Closed string

There are two possible periodicity conditions: $\psi_\pm(\sigma) = \pm \psi(\sigma + \pi)$ which can be applied separately to left & right movers

For right movers:

$$\underbrace{\psi_-^\dagger(\sigma, \tau) = \sum_{n \in \mathbb{Z}} d_n^\dagger e^{-2in(\tau-\sigma)}}_R \quad \text{or} \quad \underbrace{\psi_+^\dagger(\sigma, \tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n^\dagger e^{-2in(\tau-\sigma)}}_{NS}$$

and for left movers,

$$\underbrace{\psi_+^\dagger(\sigma, \tau) = \sum_{n \in \mathbb{Z}} \tilde{d}_n^\dagger e^{-2in(\tau+\sigma)}}_R \quad \text{or} \quad \underbrace{\psi_-^\dagger(\sigma, \tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_n^\dagger e^{-2in(\tau+\sigma)}}_{NS}$$

4 sectors: $\underbrace{NS-NS, R-R}_{\text{S.F. bosons}}, \underbrace{R-NS, NS-R}_{\text{S.F. fermions}}$

R-sector:

$$T_{\alpha\beta} = 2\alpha' \dot{x}_\mu \dot{x}_\nu \eta^{\mu\nu} + \frac{1}{2} \dot{\psi}^\dagger \rho_\beta \partial_\alpha \psi$$

$$\chi^\dagger(\sigma, \tau) = \chi^\dagger + \frac{1}{2} \rho^\beta \tau + i \frac{1}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\dagger e^{-in\tau} \cos(n\sigma), \quad \sigma^\dagger = \tau + \sigma, \quad \sigma = \tau - \sigma \rightarrow \tau = \frac{1}{2}(\sigma^\dagger + \sigma), \sigma = \frac{1}{2}(\sigma^\dagger - \sigma)$$

$$= \chi^\dagger + \frac{1}{2} \rho^\beta \tau + i \frac{1}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\dagger e^{-in(\sigma^\dagger + \sigma)/2} \cos(\frac{n}{2}(\sigma^\dagger - \sigma))$$

$$2\alpha' \dot{x}^\dagger = \frac{1}{2} \dot{\chi}^\dagger \rho^\dagger + i \frac{1}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\dagger (-\frac{i}{2} n) e^{-in(\sigma^\dagger + \sigma)/2} \cos(\frac{n}{2}(\sigma^\dagger - \sigma)) - \frac{1}{n} \alpha_n^\dagger e^{-in(\sigma^\dagger + \sigma)/2} \sin(\frac{n}{2}(\sigma^\dagger - \sigma)) \frac{n}{2}$$

$$= \frac{1}{2} \dot{\chi}^\dagger \rho^\dagger + \frac{1}{2} \sum_{n \neq 0} \alpha_n^\dagger e^{-in(\sigma^\dagger + \sigma)/2} \left(\underbrace{\cos(-) - i \sin(-)}_{e^{-in(\sigma^\dagger - \sigma)/2}} \right) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_n^\dagger e^{-in\sigma}$$

$$\alpha_n \cdot \partial_n \cdot X = \frac{1}{4} \sum_{n, m \in \mathbb{Z}} \alpha_n \cdot \alpha_m e^{-im\sigma + i n \sigma} \stackrel{m \rightarrow m-n}{=} \frac{1}{4} \sum_n \sum_m \alpha_{m-n} \cdot \alpha_n e^{-im\sigma + i n \sigma - i n \sigma} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(\frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \right) e^{-im\sigma}$$

$$\phi_n^\dagger = \frac{1}{\sqrt{2}} \sum_{m \in \mathbb{Z}} d_m^\dagger e^{-im\sigma}$$

$$\partial_n \phi_n^\dagger = \frac{1}{\sqrt{2}} \sum_{m \in \mathbb{Z}} d_m^\dagger (-im) e^{-im\sigma}$$

$$\begin{aligned} \frac{1}{2} \phi_n^\dagger \partial_n \phi_n &= \frac{1}{4} \sum_{n, m \in \mathbb{Z}} d_n \cdot d_m m e^{-im\sigma + i n \sigma} = \frac{1}{4} \sum_{n, m \in \mathbb{Z}} d_n \cdot d_{m-n} (m-n) e^{-im\sigma} \stackrel{n \rightarrow m-n}{=} -\frac{1}{4} \sum_{n, m \in \mathbb{Z}} d_{m-n} \cdot d_n (m-n) e^{-im\sigma} = -\frac{1}{4} \sum_{n \in \mathbb{Z}} d_n \cdot d_{m-n} n e^{-im\sigma} \\ &= \frac{1}{2} \left(\frac{1}{4} \sum_{n, m \in \mathbb{Z}} d_n \cdot d_{m-n} e^{-im\sigma} (m-n-n) \right) = \frac{1}{4} \sum_{n, m \in \mathbb{Z}} d_n \cdot d_{m-n} \left(\frac{m}{2} - n \right) e^{-im\sigma} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(\frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\frac{m}{2} - n \right) d_n \cdot d_{m-n} \right) e^{-im\sigma} \end{aligned}$$

$$= \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(\underbrace{\frac{1}{2} \sum_{n \in \mathbb{Z}} d_n \cdot d_{m-n}}_{L_m^{(0)}} \right) e^{-im\sigma} \quad | \quad L_m^{(0)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_n \cdot \alpha_{m-n} \quad \& \quad L_m^{(0)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(n + \frac{m}{2} \right) d_n \cdot d_{m-n}$$

$$L_0^{(0)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_n \cdot \alpha_n = \frac{1}{2} \alpha_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (\underbrace{\alpha_n \cdot \alpha_n + \alpha_{-n} \cdot \alpha_{-n}}_{[\alpha_n^\mu, \alpha_n^\mu] = m \eta^{\mu\nu} \delta_{nm,0}}) = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_n \cdot \alpha_n + \frac{1}{2} \sum_{n=1}^{\infty} n \eta^{\mu\nu} \eta_{\mu\nu}$$

$$L_0^{(0)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} n d_n \cdot d_n = \frac{1}{2} \sum_{n=1}^{\infty} (n d_n \cdot d_n - n d_{-n} \cdot d_{-n}) = \sum_{n=1}^{\infty} n d_n \cdot d_n - \frac{1}{2} \sum_{n \neq 0} n \eta^{\mu\nu} \eta_{\mu\nu}$$

$[d_n^\mu, d_n^\mu] = \eta^{\mu\nu} \delta_{nm,0}$

$$\Rightarrow \frac{1}{2} L_0^{(0)} \eta^\mu = \sum_{n=1}^{\infty} (\alpha_n \cdot \alpha_n + n d_n \cdot d_n) = N^{(0)} + N^{(0)}$$

NS-sector

$$\phi_n^\dagger(s, \sigma) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\dagger e^{-ir\sigma}, \quad \partial_n \phi_n^\dagger = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\dagger (-ir) e^{-ir\sigma}$$

$$\begin{aligned} \frac{1}{2} \phi_n^\dagger \partial_n \phi_n &= \frac{1}{4} \sum_{r, s \in \mathbb{Z} + \frac{1}{2}} s b_r^\dagger b_s \eta_{rs} e^{-is\sigma + ir\sigma} \stackrel{r \rightarrow r-s+1/2}{=} \frac{1}{4} \sum_{r, s \in \mathbb{Z} + \frac{1}{2}} s b_{r-s+1/2} \cdot b_s e^{-i(s+1/2)\sigma} \stackrel{r+s=1/2}{=} \frac{1}{4} \sum_{m \in \mathbb{Z}} \sum_{s \in \mathbb{Z} + \frac{1}{2}} s b_{m-s} \cdot b_s e^{-im\sigma} \stackrel{s \rightarrow m-s}{=} \frac{1}{4} \sum_{m \in \mathbb{Z}} \underbrace{(m-s) b_s \cdot b_{m-s}}_{-b_{m-s} \cdot b_s} e^{-im\sigma} \\ &= \frac{1}{8} \sum_{m \in \mathbb{Z}} (s+s-m) b_{m-s} \cdot b_s e^{-im\sigma} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(\frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} (s - \frac{m}{2}) b_{m-s} \cdot b_s \right) e^{-im\sigma} \end{aligned}$$

$$L_0^{(0)} = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} (r b_r \cdot b_r) = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r b_{-r} \cdot b_r + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r b_r \cdot b_{-r} = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r b_{-r} \cdot b_r - \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r b_r \cdot b_{-r} = \sum_{r \in \mathbb{Z} + \frac{1}{2}} r b_r \cdot b_r - \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r \eta^{\mu\nu} \eta_{\mu\nu}$$

$$\Rightarrow \frac{1}{2} L_0^{(0)} \eta^\mu = N^{(0)} + N^{(0)} + \frac{1}{2} \sum_{r=1}^{\infty} r \delta_r^\mu - \frac{1}{2} \sum_{r=1/2}^{\infty} r \delta_r^\mu$$

$$= \frac{1}{2} \delta_r^\mu \left(\sum_{n=1}^{\infty} n - \sum_{n=1/2}^{\infty} n \right) = \frac{1}{2} \delta_r^\mu \left(\sum_{n=1}^{\infty} n - \sum_{n=1}^{\infty} (n - \frac{1}{2}) \right) = \frac{1}{4} \delta_r^\mu \sum_{n=1}^{\infty} 1 = \frac{1}{4} \delta_r^\mu \zeta(0) = -\frac{1}{8} \delta_r^\mu = -\frac{D-2}{8} = -\frac{D-2}{16} \quad \times \quad -\frac{D-2}{16}$$

(4.6) Verify the constants $a_0=0$ and $a_{m3}=\frac{1}{2}$.

- 4.3 Consider the RNS string in 10d Minkowski space-time. Show that after the GSO projection the NS & R sectors have the same number of physical degrees of freedom at the 2nd massive level. Determine the explicit form of the states in the light-cone gauge.

Notes: Light-cone gauge

NS sector

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r \tilde{b}_{-r}^i \tilde{b}_r^i - \frac{1}{2}, \quad \alpha_n^i |0, k\rangle = \tilde{b}_r^i |0, k\rangle = 0 \quad n, r > 0$$

$$\alpha_0^i |0, k\rangle = \sqrt{2\alpha'} k^i |0, k\rangle$$

$$\alpha' M^2 |0, k\rangle = \sum_{n=1}^{\infty} \underbrace{\alpha_{-n}^i \alpha_n^i}_{=0} |0, k\rangle + \sum_{r=1/2}^{\infty} r \underbrace{\tilde{b}_{-r}^i \tilde{b}_r^i}_{=0} |0, k\rangle - \frac{1}{2} |0, k\rangle = -\frac{1}{2} |0, k\rangle \quad \text{TACHYON.}$$

Let excited state: $\tilde{b}_{-1/2}^i |0, k\rangle$ is def. $\Rightarrow$ massless vector of $SO(8)$

$$\Rightarrow \alpha' M^2 (\tilde{b}_{-1/2}^i |0, k\rangle) = 0$$

$$= \sum \alpha_{-n}^i \alpha_n^i \tilde{b}_{-1/2}^i |0, k\rangle + \sum_r \tilde{b}_{-r}^i \tilde{b}_r^i \tilde{b}_{-1/2}^i |0, k\rangle - \alpha_{1/2}^i \tilde{b}_{-1/2}^i |0, k\rangle$$

$$= \sum_r \tilde{b}_{-r}^i \tilde{b}_r^i \tilde{b}_{-1/2}^i |0, k\rangle - \alpha_{1/2}^i \tilde{b}_{-1/2}^i |0, k\rangle$$

$$\Rightarrow \frac{1}{2} - \alpha_{1/2}^i = 0 \quad \checkmark \quad = \frac{1}{2} |0, k\rangle$$

R sector

$$\alpha' M^2 = \sum_n \alpha_{-n}^i \alpha_n^i + \sum_n d_{-n}^i d_n^i \quad \alpha_n^i |0, k\rangle = d_n^i |0, k\rangle = 0 \quad n > 0$$

$[1000]$
 $\frac{1}{8}$
 $\frac{2}{2} = 8$



IB: $\left(\begin{smallmatrix} \text{NS} \\ [1000] \\ \frac{1}{8} \end{smallmatrix} \right) \left(\begin{smallmatrix} \text{R} \\ [0001] \\ \frac{1}{8} \end{smallmatrix} \right) = \left(\begin{smallmatrix} \text{NS} \\ [1000] \\ \frac{1}{8} \end{smallmatrix} \right) \left(\begin{smallmatrix} \text{R} \\ [0001] \\ \frac{1}{8} \end{smallmatrix} \right)$

$$= \left(\begin{smallmatrix} [2000] \\ 35 \end{smallmatrix} \right) + \left(\begin{smallmatrix} [0100] \\ 35 \end{smallmatrix} \right) + \left(\begin{smallmatrix} [0010] \\ 35 \end{smallmatrix} \right) + 2 \left(\begin{smallmatrix} [1001] \\ 56 \end{smallmatrix} \right) + 2 \left(\begin{smallmatrix} [0010] \\ 56 \end{smallmatrix} \right) + \left(\begin{smallmatrix} [0002] \\ 28 \end{smallmatrix} \right) + \left(\begin{smallmatrix} [0000] \\ 1 \end{smallmatrix} \right) + \left(\begin{smallmatrix} [0100] \\ 28 \end{smallmatrix} \right)$$

$\frac{1}{2} N[1000] = \frac{1}{2} \frac{8!}{2! 6!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{2 \cdot 4 \cdot 3 \cdot 2} = 35$

Graviton 2-form dilaton Gravitino Spinor Sol 4-form axion 2-form

NS-NS NS-R

Full spectrum... tachyons... no space-time zero.

To turn this into a proper theory, we need to project the states (GSO projection)

G-parity operator (NS sector): $G = (-1)^{F+1} = (-1)^{\sum_{n=1}^{\infty} \tilde{b}_{-n}^i \tilde{b}_n^i + 1}$

(R sector): $G = \Gamma_8 (-1)^{\sum_{n=1}^{\infty} d_{-n}^i d_n^i}$

GSO projection = $\begin{cases} +ve \text{ of parity in NS sector} \\ +ve/-ve \text{ " " R sector} \end{cases}$

depending on ground state spinor chirality (convention)

Gliozzi, Schemm, Olive

$\rightarrow$ 10d equivalent of Dirac matrix γ_5

$$w/ \quad \Gamma_8 = \Gamma_7 \Gamma_8 \dots \Gamma_1, \quad (\Gamma_i)^2 = 0, \quad \{\Gamma_i, \Gamma_j\} = 0, \quad \Gamma_8 \psi = \pm \psi$$

chirality

$$P_{\pm} = \frac{1}{2} (1 \pm \Gamma_8)$$

NS sector vev: $(-1)^{F+1} |0, k\rangle = (-1)^{0+1} |0, k\rangle = -|0, k\rangle \quad \times \rightarrow$ need odd number of $\tilde{b}$

$(-1)^{F+1} \tilde{b}_{-1/2}^i |0, k\rangle = (-1)^1 \tilde{b}_{-1/2}^i |0, k\rangle = \tilde{b}_{-1/2}^i |0, k\rangle \quad \checkmark$

Exercise 4.10:

NS:

level 0: GSO projected away

$\frac{1}{2}$: massless

1: GSO p. a.

R sector

levels:

0: massless $|+\rangle, |-\rangle$: $G|+\rangle = \bar{L}_0(-1)^0 |+\rangle = |+\rangle \checkmark$ $G|-\rangle = \bar{L}_0(-1)^0 |-\rangle = -|-\rangle \times$
 1: $G|\alpha_{-1}^i|+\rangle = \bar{L}_0(-1)^0 \alpha_{-1}^i|+\rangle = |+\rangle \checkmark$ $G|\alpha_{-1}^i|-\rangle = -|-\rangle \times$
 $G|d_{-1}^i|+\rangle = \bar{L}_0(-1)^0 d_{-1}^i|+\rangle = -|+\rangle \times$ $\checkmark$

NS

For $N = \frac{3}{2}$: $\frac{\alpha_{-1}^i b_{-1/2}^j |0\rangle}{8 \cdot 8 = 64}$, $\frac{b_{-1/2}^j b_{-1/2}^k b_{-1/2}^l |0\rangle}{\bar{L}_0^3(8) = \left(\frac{3}{2}\right)^3 = \frac{8 \cdot 3 \cdot 2}{2 \cdot 2} = 56}$, $\frac{b_{-3/2}^i |0\rangle}{8}$
 total = $\frac{64}{56} = \frac{8}{7}$

R

For $N=1$: $\frac{\alpha_{-1}^i |+\rangle}{8 \cdot 8 = 64}$, $\frac{d_{-1}^i |-\rangle}{8 \cdot 8 = 64}$ total = 128

NS sector, $N=5/2$: $\frac{b_{-1/2}^i b_{-1/2}^j b_{-1/2}^k b_{-1/2}^l |0\rangle}{\left(\frac{5}{2}\right)^4 = 56}$, $\frac{\alpha_{-1}^i b_{-1/2}^j b_{-1/2}^k b_{-1/2}^l |0\rangle}{8 \cdot 56 = 448}$, $\frac{\alpha_{-1}^i \alpha_{-1}^j b_{-1/2}^k |0\rangle}{\left(\frac{5}{2}\right)^3 = 8}$, $\frac{\alpha_{-1}^i b_{-1/2}^j |0\rangle}{\frac{\left(\frac{5}{2}\right)^2}{2} \cdot 8 = 288}$, $\frac{b_{-1/2}^i b_{-1/2}^j b_{-1/2}^k b_{-3/2}^l |0\rangle}{8 \cdot \left(\frac{5}{2}\right)^3 = 224}$, $\frac{\alpha_{-1}^i b_{-1/2}^j b_{-3/2}^l |0\rangle}{64}$, $\frac{b_{-5/2}^i |0\rangle}{8}$
 Total: 1152 = 128 * 9

R sector, $N=2$: $\frac{\alpha_{-1}^i \alpha_{-1}^j |+\rangle}{\left(\frac{5}{2}\right)^2 \cdot 8 = \frac{7 \cdot 5}{2} \cdot 8 = 36 \cdot 8 = 288}$, $\frac{\alpha_{-1}^i |+\rangle}{8 \cdot 8 = 64}$, $\frac{\alpha_{-1}^i d_{-1}^j |-\rangle}{8 \cdot 8 \cdot 8 = 512}$, $\frac{d_{-1}^i d_{-1}^j |+\rangle}{\left(\frac{5}{2}\right)^2 \cdot 8 = 28 \cdot 8 = 224}$, $\frac{d_{-1}^i |+\rangle}{8 \cdot 8 = 64}$
 Total: 1152

4.8 Given a pair of 2d Majorana spinors ψ & $\bar{\psi}$ prove that $\psi_a \bar{\psi}_b = -\frac{1}{2} (\bar{\psi} \psi \delta_{ab} + \bar{\psi} \rho_a \psi \rho_{ab} + \bar{\psi} \rho_3 \psi (\rho_3)_{ab})$.

Want to construct a complete orthogonal basis. Consider

$\vec{r}^1: \underline{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{r}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\vec{r}^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{r}^4 = \vec{r}^1 \vec{r}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\vec{r}^5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{r}^6 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\vec{r}^7 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{r}^8 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ } clearly complete

Check orthogonality: $\text{Tr}(\vec{r}^i \vec{r}^j) = 2\delta^{ij}$

Trivially, if $I=J$ $\text{Tr}(\vec{r}^i \vec{r}^i) = 2$

If $I \neq J$, $\text{Tr} \underline{1} \vec{r}^j = \text{Tr} \underline{1} \vec{r}^j = \text{Tr} \underline{1} \vec{r}^j = 0$

$\text{Tr} \vec{r}^i \vec{r}^j = \text{Tr} \vec{r}^j \vec{r}^i = 0$

$\text{Tr} \vec{r}^i \vec{r}^j = 0$ $\checkmark$

Any matrix can be written as $M_{AB} = \sum_{I=1}^8 c_I (\vec{r}^I)_{AB} \Rightarrow \sum_{A,B} M_{AB} \vec{r}^I_{AB} = \sum_{I=1}^8 c_I \sum_{A,B} (\vec{r}^I)_{AB} (\vec{r}^I)_{AB} = 2c_I \Rightarrow c_I = \frac{1}{2} \sum_{A,B} M_{AB} \vec{r}^I_{AB}$
 $\Rightarrow M_{AB} = \sum_{I=1}^8 \frac{1}{2} \sum_{C,D} M_{CD} \vec{r}^I_{CD} \vec{r}^I_{AB} = \sum_{C,D} M_{CD} \left(\frac{1}{2} \sum_{I=1}^8 (\vec{r}^I)_{CD} (\vec{r}^I)_{AB} \right) = \delta_{CA} \delta_{DB}$

$$\phi_A \bar{\phi}_B = \phi_C \bar{\phi}_D (\delta_{AC} \delta_{DB}) = \phi_C \bar{\phi}_D \left(\frac{1}{2} \sum_{\vec{r}} (\vec{r}^z)_{AB} (\vec{r}^z)_{DC} \right) = \frac{1}{2} \left(\phi_C \bar{\phi}_D \delta_{AB} \delta_{DC} + \bar{\phi}_D (\rho_{DC})_A \phi_C \rho_{AB}^A + \bar{\phi}_D (\rho_{DC})_{DC} \phi_C (\rho_{AB})_{AB} \right)$$

4.9) Derive the NS-sector Lorentz transformation generators in the light-cone gauge.

4.10) Show that Eqs. (4.131) and (4.133) lead to the mode expansions of L_n and G_r given earlier.

NS sector:

$$(4.131) \quad T_0(z) = -2\alpha' \dot{X}^\mu(z) \cdot \partial X_\mu(z) - \frac{1}{2} \dot{\psi}^\mu(z) \partial \psi_\mu(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \quad \dagger \quad L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_{m+n} + \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(r_{-n}^{\frac{m}{2}} \right) b_{-n} \cdot b_{m+n}$$

$$(4.133) \quad T_F(z) = 2i\psi^\mu(z) \partial \psi_\mu(z) = \sum_{r \in \mathbb{Z}} \frac{G_r}{z^{r+2}} \quad \dagger \quad G_r = \frac{i}{\pi} \oint_{\gamma} e^{i\sigma} J_r = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n}$$

$$\text{RNS string action: } S = -\frac{1}{2\pi\alpha'} \int d\tau \left(\partial_\mu X^\mu \partial_\nu X^\nu + \bar{\psi}^\mu \rho^\mu \partial_\nu \psi_\nu \right)$$

$$\text{Wick rotation: } \tau = -i\bar{\tau} \Rightarrow d\tau = -i d\bar{\tau}, \quad \frac{\partial}{\partial \tau} = \frac{\partial \bar{\tau}}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} = i \frac{\partial}{\partial \bar{\tau}}$$

$$S = + \frac{i}{2\pi} \int d\sigma d\bar{\tau} \left(-i^2 \partial_{\bar{\tau}} X^\mu \partial_{\bar{\tau}} X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu - i \bar{\psi}^\mu \rho^\mu \partial_{\bar{\tau}} \psi_\mu + \bar{\psi}^\mu \rho^\mu \partial_\sigma \psi_\mu \right)$$

$$z = e^{2i(\bar{\tau}-i\sigma)}, \quad \bar{z} = e^{2i(\bar{\tau}+i\sigma)}, \quad \bar{\tau}-i\sigma = \frac{1}{2} \ln(\bar{z}), \quad \bar{\tau}+i\sigma = \frac{1}{2} \ln(z) \Rightarrow 2i\bar{\tau} = \frac{1}{2} \ln \bar{z} + \frac{1}{2} \ln z = \ln \sqrt{\bar{z}z} \quad \dagger \quad 2i\sigma = \frac{1}{2} \ln \bar{z} - \frac{1}{2} \ln z = \ln \sqrt{\frac{\bar{z}}{z}}$$

$$\bar{\tau} = \frac{1}{4} \ln \bar{z} \quad \dagger \quad \sigma = \frac{i}{4} \ln \frac{\bar{z}}{z}$$

$$dz = 2z d\bar{\tau} - 2iz d\sigma \quad \dagger \quad d\bar{z} = 2\bar{z} d\bar{\tau} + 2i\bar{z} d\sigma$$

$$\frac{dz}{2z} = d\bar{\tau} - i d\sigma \quad \dagger \quad \frac{d\bar{z}}{2\bar{z}} = d\bar{\tau} + i d\sigma \Rightarrow d\bar{\tau} = \frac{1}{4} \left(\frac{dz}{z} + \frac{d\bar{z}}{\bar{z}} \right) \quad \dagger \quad d\sigma = \frac{i}{4} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \Rightarrow d\sigma = d\bar{\tau} \wedge d\sigma = \frac{i}{16} \left(\frac{1}{z\bar{z}} dz \wedge d\bar{z} - \frac{1}{z\bar{z}} d\bar{z} \wedge dz \right)$$

$$d\bar{\tau} = \frac{\partial \bar{\tau}}{\partial z} dz + \frac{\partial \bar{\tau}}{\partial \bar{z}} d\bar{z} = \frac{1}{4} dz + \frac{1}{4} d\bar{z}$$

$$d\sigma = -\frac{1}{4} dz + \frac{1}{4} d\bar{z}$$

$$d\sigma \wedge d\bar{\tau} = \frac{1}{16} dz \wedge d\bar{z} = -\frac{i}{8} \frac{1}{z\bar{z}} dz \wedge d\bar{z}$$

$$\partial_{\bar{\tau}} X^\mu \cdot \partial_{\bar{\tau}} X_\mu = 4 \left(z \partial_X \cdot \bar{z} \partial_X \right) \cdot \left(\bar{z} \partial_X + z \partial_X \right) = 4 \left(z^2 \partial_X \cdot \partial_X + 2z\bar{z} \partial_X \cdot \partial_X + \bar{z}^2 \partial_X \cdot \partial_X \right)$$

$$\partial_\sigma X^\mu \cdot \partial_\sigma X_\mu = -4 \left(z \partial_X - \bar{z} \partial_X \right) \cdot \left(z \partial_X - \bar{z} \partial_X \right) = -4 \left(z^2 \partial_X \cdot \partial_X - 2z\bar{z} \partial_X \cdot \partial_X + \bar{z}^2 \partial_X \cdot \partial_X \right)$$

$$-i \bar{\psi}^\mu \rho^\mu \partial_{\bar{\tau}} \psi_\mu = \bar{\psi}^\mu \rho^\mu \partial_{\bar{\tau}} \psi_\mu = (\bar{\psi}^\mu \psi^\mu) \cdot (2z\partial + 2\bar{z}\partial) \left(\frac{\psi^\mu}{\psi^\mu} \right) = 2z \bar{\psi}^\mu \partial \psi^\mu + 2\bar{z} \bar{\psi}^\mu \partial \psi^\mu + 2z \bar{\psi}^\mu \partial \psi^\mu + 2\bar{z} \bar{\psi}^\mu \partial \psi^\mu$$

$$\bar{\psi}^\mu \rho^\mu \partial_\sigma \psi_\mu = (\bar{\psi}^\mu \psi^\mu) \cdot \left(\frac{1}{4} \frac{dz}{z} - \frac{1}{4} \frac{d\bar{z}}{\bar{z}} \right) \cdot (-2iz\partial + 2i\bar{z}\partial) \left(\frac{\psi^\mu}{\psi^\mu} \right)$$

$$= 2(\bar{\psi}^\mu \psi^\mu) (z\partial - \bar{z}\partial) \left(\frac{\psi^\mu}{\psi^\mu} \right) = -2z \bar{\psi}^\mu \partial \psi^\mu + 2\bar{z} \bar{\psi}^\mu \partial \psi^\mu + 2z \bar{\psi}^\mu \partial \psi^\mu - 2\bar{z} \bar{\psi}^\mu \partial \psi^\mu$$

$$\Rightarrow S = \frac{i}{2\pi} \int d\bar{\tau} \frac{1}{8} \frac{1}{z\bar{z}} \left(16z\bar{z} \partial_X \partial_X + 4z\bar{z} \bar{\psi}^\mu \partial \psi^\mu + 4z\bar{z} \bar{\psi}^\mu \partial \psi^\mu \right) = \frac{i}{2\pi} \int d\bar{\tau} \left(2\partial_X \cdot \partial_X + \frac{1}{2} \bar{\psi}^\mu \partial \psi^\mu + \frac{1}{2} \bar{\psi}^\mu \partial \psi^\mu \right)$$

Conformal transformation:
 $0(z, \bar{z}) \rightarrow 0(\bar{z}, \bar{z}) = \left(\frac{\partial \bar{z}}{\partial z} \right)^h \left(\frac{\partial \bar{z}}{\partial \bar{z}} \right)^{\bar{h}} 0$
 $SO = -h(\partial)0 - \bar{h}\partial 0$

$$\text{For } z \rightarrow z, \bar{z} \rightarrow \bar{z}, \quad SS = \frac{i}{2\pi} \int d\bar{\tau} \left(2\partial_X \cdot \partial_X + 2\partial_X \cdot \partial_X + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu \right)$$

$$= \int d\bar{\tau} \left(2\partial_X \cdot \partial_X + 2\partial_X \cdot \partial_X + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu \right)$$

$$= \int d\bar{\tau} \left(2\partial_X \cdot \partial_X + 2\partial_X \cdot \partial_X + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu \right)$$

$$= \int d\bar{\tau} \left(2\partial_X \cdot \partial_X + 2\partial_X \cdot \partial_X + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu + \frac{1}{2} (\bar{\psi}^\mu) \cdot \partial \psi^\mu \right)$$

$$\begin{aligned}
 &= \left(\partial_z^2 - \frac{1}{2} \partial_z (\partial_z \bar{\partial} \phi) + \frac{1}{4} (\partial_z^2 \phi) \cdot \bar{\partial} \phi - \frac{1}{4} (\bar{\partial}^2 \bar{z}) \phi \cdot \bar{\partial} \phi - \frac{1}{4} (\partial_z \phi) \cdot \bar{\partial} \phi - \frac{1}{2} (\bar{\partial} z) \phi \cdot \bar{\partial} \phi - \frac{1}{2} \partial_z (\bar{z} \phi \cdot \bar{\partial} \phi) \right) \\
 &= \left(\partial_z^2 (\bar{\partial} z) - \underbrace{2 \partial_z \bar{\partial} \phi - \frac{1}{2} \phi \cdot \bar{\partial} \phi}_{= T_B(z)} \right)
 \end{aligned}$$

$$\begin{aligned}
 T_{\text{CG}}(h) &= \sum_{n \in \mathbb{Z}} L_n e^{-2\pi n}, \quad z = e^h \\
 &= \left(\frac{\partial z}{\partial h} \right)^h T_{\text{plane}}(z) = e^{2\pi h} T_{\text{plane}}(z) = z^{2\pi} T_{\text{plane}}(z)
 \end{aligned}$$

$$T_{\text{plane}}(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}$$

Susy transformations:

$$\begin{aligned}
 \delta \chi^I &= \bar{z} \phi^I = (z, z_*) i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_z \\ \phi_{z_*} \end{pmatrix} = -i z \cdot \bar{\psi} + i z_* \phi \\
 \delta \phi^I &= \bar{\psi}^I \partial_\mu \chi^I z = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_\mu \chi \begin{pmatrix} z \\ z_* \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\mu \chi \begin{pmatrix} z_* \\ z \end{pmatrix} \\
 &= \begin{pmatrix} -i(\partial_z + \bar{\partial}) \chi z_* \\ i(\partial_z + \bar{\partial}) \chi z \end{pmatrix} + \begin{pmatrix} i(-\partial_z + \bar{\partial}) \chi z_* \\ i(\partial_z + \bar{\partial}) \chi z \end{pmatrix} \\
 &= \begin{pmatrix} -2i \partial_z \chi z_* \\ 2i \partial_z \chi z \end{pmatrix}
 \end{aligned}$$

$$\tau = -i\bar{z} \quad \partial_z = \frac{\partial \bar{z}}{\partial \tau} \partial_\tau = i \partial_\tau$$

$$z = \tau + i\sigma, \quad \bar{z} = \tau - i\sigma$$

$$\tau = \frac{1}{2}(z + \bar{z}), \quad \sigma = \frac{i}{2}(z - \bar{z})$$

$$\partial_z = \left(\frac{\partial \tau}{\partial z} \right) \partial + \left(\frac{\partial \sigma}{\partial z} \right) \bar{\partial} = \partial + \bar{\partial} \quad \partial_{\bar{z}} = -i \partial + i \bar{\partial}$$

$$\delta_z \chi^I = i z_* \phi^I \quad \delta \phi^I = -2i z_* \partial \chi$$

$$\begin{aligned}
 \delta_z S &\sim \int d^2 z \left(2 \partial_z (\delta \chi) \cdot \bar{\partial} \chi + 2 \partial_z \bar{\partial} (\delta \chi) + \frac{1}{2} (\delta \phi) \bar{\partial} \phi + \frac{1}{2} \phi \bar{\partial} (\delta \phi) + \frac{1}{2} \bar{\psi} \partial \bar{\psi} + \frac{1}{2} \bar{\psi} \partial (\delta \bar{\psi}) \right) \\
 &= \int d^2 z \left(2 \partial_z (i z_* \phi) \cdot \bar{\partial} \chi + 2 \partial_z \bar{\partial} (i z_* \phi) + \frac{1}{2} (-2i z_* \partial \chi) \cdot \bar{\partial} \phi + \frac{1}{2} \phi \bar{\partial} (-2i z_* \partial \chi) \right) \\
 &= \int d^2 z \left(\underbrace{2i (\partial_z z_*) \phi \cdot \bar{\partial} \chi + 2i z_* \partial \phi \cdot \bar{\partial} \chi + 2i (\bar{\partial} z_*) \phi \cdot \partial \chi + 2i z_* \bar{\partial} \phi \cdot \partial \chi}_{-2i z_* \phi \cdot \partial \chi = 0?} - \underbrace{i z_* \bar{\partial} \phi \cdot \partial \chi}_{T_B(z)} - \underbrace{i (\bar{\partial} z_*) \phi \cdot \partial \chi + i z_* \phi \cdot \bar{\partial} \partial \chi}_{-i (\bar{\partial} z_*) \phi \cdot \partial \chi - i z_* \bar{\partial} \phi \cdot \partial \chi} \right)
 \end{aligned}$$

$$f(z) = \sum a_n z^n, \quad a_n = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z^{n+1}} \rightarrow f(z) = \sum \frac{1}{(n!)} \frac{d^n}{dz^n} (f(z)) z^n \text{ Taylor}$$

$$T_B = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \Rightarrow L_n = \frac{1}{2\pi i} \oint dz \frac{T_B(z)}{z^{n+1}}$$

$$\begin{aligned}
 T_B(z) &= -2 \partial_z \bar{\partial} \chi - \frac{1}{2} \phi \cdot \partial \phi \quad \chi^I(z) = z^I + \frac{1}{2} \rho^I \tau + i \frac{1}{2} \sum_{n \neq 0} \frac{1}{n!} \partial_\mu \chi^I e^{-in\tau} \cos(n\sigma) \\
 &\quad - i \frac{1}{2} \rho^I \bar{z} \quad \text{Wick} \rightarrow \frac{1}{2} (e^{-in(\tau-\sigma)} + e^{in(\tau+\sigma)}) \\
 &\quad \rightarrow \frac{1}{2} (e^{-in(\tau-i\sigma)} + e^{-in(\tau+i\sigma)}) = \frac{1}{2} (z^{-n} + \bar{z}^{-n}) \quad \text{w/ } z = e^{\tau-i\sigma} \rightarrow \tau-i\sigma = \ln(z) \\
 &\quad \tau+i\sigma = \ln(\bar{z})
 \end{aligned}$$

$$\chi^I(z) = z^I + \frac{1}{2} \rho^I \tau + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n!} \alpha_n^I (z^{-n} + \bar{z}^{-n})$$

$$\partial \chi^I(z) = \frac{1}{2} \rho^I \alpha_0^I z^{-1} - \frac{i}{2} \sum_{n \neq 0} \alpha_n^I z^{-n-1} = -\frac{i}{2} \sum_{n \in \mathbb{Z}} \alpha_n^I z^{-n-1}$$

NS:

$$\phi_r^I(z) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n^I e^{-in(\tau-\sigma)} \xrightarrow{\text{Wick}} \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} b_n^I e^{-in(\tau-i\sigma)} = \frac{1}{\sqrt{2}} \sum_r b_r^I z^{-r} = \phi_r^I(z) \quad \dagger \quad \partial \phi_r^I = \frac{1}{\sqrt{2}} \sum_s r b_s^I z^{-r-s-1}$$

$$T_B(z) = -2 \partial_z \bar{\partial} \chi - \frac{1}{2} \phi \cdot \partial \phi$$

$$\begin{aligned}
 L_n &= \frac{1}{2\pi i} \oint dz \omega^{n+1} \left(-2 \cdot \frac{-\rho^I}{4} \sum_{m,n} \alpha_m \cdot \alpha_n \omega^{-m-n-2} + \frac{1}{2} \sum_{r,s} i b_r \cdot b_s i s \omega^{-r-s-1} \right) \\
 &= \frac{1}{2\pi i} \oint dz + \frac{\rho^I}{2} \sum_{m,n} \alpha_m \cdot \alpha_n \omega^{k-m-n-1} + \frac{1}{4} \sum_{r,s} i b_r \cdot b_s i s \omega^{k-r-s}
 \end{aligned}$$

$$\text{For } k-m-n-1 \geq 0, \quad L_k = 0$$

$$\text{For } k-r-s = -1, \quad \frac{1}{2\pi i} \oint dz \omega^k = 1$$

$$k-m-n-1 = -1, \quad k = m+n, \quad \frac{1}{2\pi i} \oint dz \omega^k = 1$$

$$\begin{aligned}
 k-m-n-1 = -2, \quad k = m+n+1, \quad \frac{1}{2\pi i} \oint dz \omega^k &= 0 \\
 \vdots & \\
 &= 0
 \end{aligned}$$

$$\Rightarrow L_4 = \frac{1}{2} \sum_{m,n} : \alpha_{-m} \cdot \alpha_n : \delta_{k+m,n} + \frac{1}{4} \sum_{r,s} : b_r \cdot b_s : \delta_{k+r,s-1}$$

$$\begin{aligned} & \xrightarrow{k \rightarrow -n} \sum_n : \alpha_{k-n} \cdot \alpha_n : = \sum_n : \alpha_{-n} \cdot \alpha_{n+k} : \\ & \sum_s : b_{k-s+1} \cdot b_s : = \sum_s : b_s \cdot b_{s+k+1} : = \sum_s : b_{s+k+1} \cdot b_{s+1} : (-s) \\ & = \frac{1}{2} \sum_s (: b_s \cdot b_{s+k+1} : (s+k+1) + : b_{s+1} \cdot b_{s+k+1} : (s)) \\ & = \frac{1}{2} \sum_s (2s+k+1) : b_{-s} \cdot b_{s+k+1} : \end{aligned}$$

$$T_F(z) = 2i\phi \cdot \partial X = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{G_r}{z^{r+3/2}}, \quad \text{for } f(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^n, \quad \alpha_n = \frac{1}{2\pi i} \oint_{\gamma} d\omega \frac{f(\omega)}{\omega^{n+1}}$$

$$r \rightarrow -r - 3/2$$

$$G_r = \frac{1}{2\pi i} \oint_{\gamma} d\omega \frac{T_F(\omega)}{\omega^{r+1/2}} = \frac{-i\beta}{2\sqrt{2}} \frac{1}{2\pi i} \oint_{\gamma} d\omega \sum_{s,n} : b_s \cdot \alpha_n : \omega^{r-s-n-1/2}$$

$$\xrightarrow{\text{for } r-s-n-1/2 = -1 \Rightarrow s = r-n+1/2}$$

$$\phi(z) = \frac{1}{\sqrt{2}} \sum_s b_s z^{-s-1/2}, \quad \partial X(z) = -\frac{i\beta}{2} \sum_n \alpha_n z^{-n-1}$$

4.8 Using T_B and T_F , verify that (4.18) holds for $\hat{c}=10$.

$$(4.18) \quad T_F(z) T_F(w) \sim \frac{\hat{c}}{4(z-w)^3} + \frac{T_F(w)}{2(z-w)} + \dots$$

$$\text{General OPE: } \mathcal{O}_1(z) \mathcal{O}_2(w) = \sum_k C_k^0(z-w, \bar{z}-\bar{w}) \mathcal{O}_k(w, \bar{w})$$

$$\begin{aligned} \partial X^\mu(z) \partial X^\nu(w) &= -\frac{\beta^2}{4} \sum_{m,n} : \alpha_{-m}^\mu : \alpha_n^\nu : z^{-m-1} w^{-n-1} = -\frac{\beta^2}{4} \sum_{m=1}^{\infty} m z^{-m-1} w^{-m-1} = -\frac{\beta^2}{4} \frac{1}{w^2} \sum_{m=1}^{\infty} m \left(\frac{z}{w}\right)^m \\ &= \sum_{m,n \in \mathbb{Z}} \langle 0 | \alpha_m^\mu \alpha_n^\nu | 0 \rangle = \sum_{\substack{n \in \mathbb{Z} \\ m \geq 1}} \langle 0 | \alpha_m^\mu \alpha_n^\nu | 0 \rangle = \sum_{\substack{n \in \mathbb{Z} \\ m \geq 0}} \langle 0 | m \delta_{m+n,0} + \alpha_m^\mu \alpha_n^\nu | 0 \rangle = \sum_{\substack{n \in \mathbb{Z} \\ m \geq 0}} m \delta_{-n,m} = \sum_{m \geq 1} m \\ &= -\frac{\beta^2}{4} \frac{1}{w^2} \frac{\frac{z^2}{4}}{(1-\frac{z}{w})^2} = -\frac{\beta^2}{4} \frac{1}{(z-w)^2} \end{aligned}$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \xrightarrow{\frac{d}{dz}} \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \xrightarrow{z} \sum_{n=0}^{\infty} n z^n$$

$$\begin{aligned} \phi(z) \phi(w) &= \frac{1}{2} \sum_{r,s} : b_{-r} : b_s : z^{-r-1/2} w^{-s-1/2} = \frac{1}{2} \sum_r z^{-r-1/2} w^{-r-1/2} = \frac{1}{2} \frac{1}{\sqrt{2}w} \sum_{r \in \mathbb{Z}} \left(\frac{z}{w}\right)^r = \frac{1}{2} \frac{1}{\sqrt{2}w} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n \left(\frac{w}{z}\right) = \frac{1}{2z} \frac{1}{1-(w/z)} = \frac{1}{2} \frac{1}{z-w} \\ &= \sum_{\substack{s \in \mathbb{Z} \\ r \geq 1/2}} : \alpha_r b_s : = \sum_{\substack{s \in \mathbb{Z} \\ r \geq 1/2}} : \alpha_r b_{s-r} : = \sum_{\substack{s \in \mathbb{Z} \\ r \geq 1/2}} : \alpha_r \delta_{s-r,0} : = \sum_{\substack{s \in \mathbb{Z} \\ r \geq 1/2}} : \alpha_r \delta_{r,s} : = \sum_{s \in \mathbb{Z}} : \alpha_s : \end{aligned}$$

$$\begin{aligned} \phi(z) \partial X(w) &= -\frac{i\beta}{2\sqrt{2}} \sum_{r,m} : b_{-r} : \alpha_m : z^{-r-1/2} w^{-m-1} \\ &= \sum_{\substack{m \in \mathbb{Z} \\ r \geq 1/2}} \langle 0 | b_r \alpha_m | 0 \rangle = \sum_{\substack{m \in \mathbb{Z} \\ r \geq 1/2}} \langle 0 | b_r \alpha_{m-r} | 0 \rangle = 0 \end{aligned}$$

$$\begin{aligned} T_F(z) T_F(w) &= -4 : \phi(z) \partial X(w) : = -4 : \phi(w) \partial X(z) : = -4 : \phi(z) \partial X(z) : = -4 : \phi(w) \partial X(w) : \\ &= -4 \frac{1}{2} \frac{1}{(z-w)^2} - 4 \frac{1}{4} \frac{1}{(z-w)^3} - 4 \frac{1}{2} \frac{1}{(z-w)} : \partial X(z) \partial X(w) : \end{aligned}$$

$$\phi(w) = \phi(z) + a \partial \phi + \frac{1}{2} \partial^2 \phi + \dots \quad \partial X(z) = \partial X(w) + (z-w) \partial^2 X(w) + \dots$$

$$\begin{aligned} \phi(z) &= \phi(w) + (z-w) \partial \phi + \frac{1}{2} (z-w)^2 \partial^2 \phi + \dots \\ &= \frac{1}{2} \frac{1}{(z-w)^2} + \frac{1}{2} \frac{1}{(z-w)^3} \left(\phi(z) + (z-w) \partial \phi + \dots \right) - 2 \frac{\partial \phi}{(z-w)} = \frac{2\eta_1}{4(z-w)^3} + \frac{1}{2(z-w)} \left(-\frac{1}{2} \phi \partial \phi - 4 \partial X \partial X \right) \end{aligned}$$

4.13 Show that $S = S_{matter} + S_{ghost}$ is BRST invariant.

$$2\partial X \cdot \partial X + 2\partial X \cdot \partial X \quad \frac{1}{2} \psi \cdot \partial \psi + \frac{1}{2} \psi \cdot \partial \psi = \psi \cdot \partial \psi$$

$$S = \frac{1}{2\pi} \int d^2 z \quad 2\partial X \cdot \partial X + \frac{1}{2} \psi \cdot \partial \psi + \frac{1}{2} \bar{\psi} \cdot \partial \bar{\psi} + b \bar{c} + \bar{b} c + \beta \bar{\gamma} + \bar{\beta} \gamma$$

$$= \frac{1}{2\pi} \int d^2 z \quad 2\partial X \cdot \partial X + \frac{1}{2} \psi \cdot \partial \psi + b \bar{c} + \beta \bar{\gamma} + \dots$$

$$\delta S = \frac{1}{2\pi} \int d^2 z \quad -4(\partial X) \cdot \partial \delta X + (\delta \psi) \cdot \partial \psi + (\delta b) \bar{c} - \bar{b} \delta c + (\delta \beta) \bar{\gamma} - \bar{\beta} \delta \gamma + \dots$$

$$= \frac{1}{2\pi} \int d^2 z \quad -4\eta c \partial X \cdot \partial \delta X + \underbrace{2\eta \partial c \partial X \cdot \partial \delta X}_{\Rightarrow -4\eta c \partial X \cdot \partial \delta X} + \underbrace{\eta c \partial \psi \cdot \partial \psi}_{\Rightarrow 2\eta \partial c \psi \cdot \partial \psi} + \frac{1}{2} \eta \partial c \psi \cdot \partial \psi + \underbrace{2i\eta \gamma \partial X \cdot \partial \psi}_{\Rightarrow \eta \bar{b} \bar{c} - \bar{b} \eta c \partial c + \bar{b} \eta \gamma \delta^t + \eta \bar{b} \bar{\gamma} \delta - \bar{\beta} \eta c \partial \gamma + \bar{\beta} \frac{1}{2} \eta \gamma \partial c}$$

$$= \frac{1}{2\pi} \int d^2 z \quad 2\eta \partial c \partial X \cdot \partial \delta X + \eta c \partial \psi \cdot \partial \psi - \frac{1}{2} \eta c \partial \psi \cdot \partial \psi - \frac{1}{2} \eta c \psi \cdot \partial \bar{c} \psi + \dots$$

$$= \frac{1}{2\pi} \int d^2 z \quad 2\eta \partial c \partial X \cdot \partial \delta X + \frac{1}{2} \eta c \partial \psi \cdot \partial \psi - \frac{1}{2} \eta c \psi \cdot \partial \bar{c} \psi + \dots$$

$$= \frac{1}{2\pi} \int d^2 z \quad 2\eta \partial c \partial X \cdot \partial \delta X + \frac{1}{2} \eta \bar{c} \psi \cdot \partial \psi + \dots$$

$$= \frac{1}{2\pi} \int d^2 z \quad -\eta \bar{b} \bar{c} + 2i\eta \gamma \psi \cdot \partial \delta X + 2i\eta \gamma \partial X \cdot \partial \psi + \eta \bar{b} \bar{c} + \dots + \eta \bar{b} (\delta \gamma)$$

$$= \frac{1}{2\pi} \int d^2 z \quad -\bar{b} \eta c \partial c + \bar{b} \eta \gamma \delta^t - \bar{b} \eta c \partial \gamma + \frac{1}{2} \bar{\beta} \eta \gamma \partial c + \eta (-2b \partial c + c \partial b - \frac{3}{2} \beta \partial \gamma - \frac{1}{2} \gamma \partial \beta) \bar{c} + \eta (-2b \gamma + c \partial \beta + \frac{3}{2} \beta \partial c) \bar{\gamma}$$

$$= \frac{1}{2\pi} \int d^2 z \quad \left(\underbrace{\eta \bar{b} c \partial c + 2\eta \bar{b} c \partial c - \eta \bar{b} c \bar{c}}_{\bar{b} c \partial c} \right) + \left(\underbrace{-\eta \bar{b} \gamma \delta^t - 2\eta \bar{b} \gamma \partial \gamma}_{\bar{b} \gamma \partial \gamma} \right) + \left(\underbrace{-\eta \bar{\beta} \partial \gamma c + \frac{1}{2} \eta \bar{\beta} \gamma \partial c - \frac{3}{2} \eta \bar{\beta} \partial \gamma \bar{c} - \frac{1}{2} \eta \bar{\beta} \gamma \partial \bar{c} + \eta \bar{\beta} \partial \gamma c}_{\frac{3}{2} \eta \bar{\beta} \partial \gamma c + \frac{3}{2} \eta \bar{\beta} \partial \gamma \bar{c} + \frac{3}{2} \eta \bar{\beta} \gamma \partial c} \right) + \left(\underbrace{-\frac{1}{2} \bar{\beta} \partial \gamma c + \frac{1}{2} \bar{\beta} \partial \gamma \bar{c} + \frac{1}{2} \bar{\beta} \gamma \partial c + \frac{1}{2} \bar{\beta} \gamma \partial \bar{c} - \frac{1}{2} \bar{\beta} \partial \gamma c - \frac{1}{2} \bar{\beta} \gamma \partial \bar{c} - \frac{1}{2} \bar{\beta} \gamma \partial c - \frac{1}{2} \bar{\beta} \gamma \partial \bar{c}}_{-\frac{1}{2} \bar{\beta} \gamma \partial c} \right) + \left(\underbrace{-\frac{1}{2} \bar{\beta} \gamma \partial c + \frac{1}{2} \bar{\beta} \gamma \partial \bar{c} + \frac{1}{2} \bar{\beta} \partial \gamma c + \frac{1}{2} \bar{\beta} \partial \gamma \bar{c} - \frac{1}{2} \bar{\beta} \gamma \partial c - \frac{1}{2} \bar{\beta} \gamma \partial \bar{c} - \frac{1}{2} \bar{\beta} \partial \gamma c - \frac{1}{2} \bar{\beta} \partial \gamma \bar{c}}_{-\frac{1}{2} \bar{\beta} \gamma \partial c} \right) + \dots$$

4.14 Consider the ghost contributions to the super-Virasoro generators in the NS sector.

(i) Work out the mode expansions for ghost contributions to L_n and G_r implied by

$$S_{ghost} = \frac{1}{2\pi} \int d^2 z \quad T_{ghost}^{\mu\nu} + T_{ghost}^{\mu\nu}$$