

5.1 Show that the action $S = -m \int dt \sqrt{-\Pi_\mu \cdot \Pi_\mu} - m \int dt \tilde{\Theta} \Gamma_\mu \dot{\Theta}$ is invariant under a reparameterisation of the world line.

The D0 brane = susy point particle

Introduce anti-commuting spinor coordinates

(Point particle in Minkowski: $S = -m \int dt \sqrt{-\dot{x} \cdot \dot{x}}$)

↳ want to generalise this

③ Pa on w/ $A=1, \dots, W$ & $a=1, \dots, 2^{D_A}$ for Deven and spinor=disc.

Define $\Pi_\alpha^r = \partial_\alpha \chi^r - \bar{\Theta}^A \Gamma^r \partial_\alpha \Theta^A \quad \alpha = 0, \dots, p$

↳ $\Pi^\dagger$ is invariant under susy, thus make susy action by $\chi^i \rightarrow \Pi^\dagger$

$S_1 = -m \int d\tau \sqrt{-\Pi_0 \cdot \Pi_0}$ invariant under: global super-Poincaré & local diffeomorphisms of worldline.

In Rd, $N=2$. For Majorana, $\Theta = \Theta' + \Theta''$ w/ $\Theta' = \frac{1}{2}(1 + \Gamma_n)\Theta$ & $\Theta'' = \frac{1}{2}(1 - \Gamma_n)\Theta$

$$\begin{aligned} \partial^* \Gamma \partial^* &= \partial^* \Gamma \partial^* \cdot \partial^* \Gamma \partial^* = \frac{1}{4} (\overline{1-\Gamma_0}) \partial^* \Gamma (1-\Gamma_0) + \frac{1}{4} (\overline{1-\Gamma_0}) \partial^* \Gamma (1-\Gamma_0) \\ &= \frac{1}{4} \left(\underbrace{\partial^* \Gamma \partial^* + \overline{\Gamma_0} \Gamma \Gamma_0}_{= \partial^* \Gamma \partial^*} + \underbrace{\partial^* \Gamma \partial^* + \overline{\Gamma_0} \Gamma \Gamma_0}_{= \partial^* \Gamma \partial^*} \right) + \frac{1}{4} \left(\underbrace{\overline{\Gamma_0} \Gamma \partial^* + \partial^* \Gamma \Gamma_0 - \overline{\Gamma_0} \Gamma \Gamma_0 - \partial^* \Gamma \Gamma_0}_{= 0} \right) \\ &\quad - (\Gamma_0) i^* \Gamma_0 \Gamma_0 = i^* \Gamma_0 \Gamma_0 \Gamma_0 \\ &= \partial^* \Gamma \partial^* = i^* \Gamma_0 \Gamma_0 \Gamma_0 \longrightarrow \Gamma_0^* = (\Gamma_0 \Gamma_0 \Gamma_0)^* = \Gamma_0^* \Gamma_0^* \Gamma_0^* = (\Gamma_0 \Gamma_0) \dots (\Gamma_0) \end{aligned}$$

Conjugate momentum to x^r is

$$P_I = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_I} = \frac{\partial}{\partial \dot{\chi}_I} \left(-m \int dt \sqrt{(\dot{\chi}^I - \partial_I r \dot{\phi}) (\dot{\chi}_I - \partial_I r \dot{\phi})} \right) \\ = \frac{1}{2} (-\Pi_0 \cdot \Pi_0)^{1/2} = 2 (\dot{\chi}_I - \partial_I r \dot{\phi})$$

and com for x^r

$$\delta S_1 = -m \int dt \frac{1}{2} (-\Pi_0 \cdot \Pi_0)^{-1/2} \cdot 2 \partial_0 (\delta X^I) (\dot{X}_I - \bar{\Theta} \Gamma_I \dot{\Theta}) \Rightarrow \dot{P}_I = 0$$

$$p_r p^r = \frac{m^2}{-\Pi_r \Pi^r} \underbrace{(\dot{x}_r - \partial_r \Gamma \dot{\theta})(\dot{x}^r - \partial^r \Gamma \dot{\theta})}_{\Pi_r \Pi^r} = -m^2 \quad \checkmark$$

Majorana: $\bar{\psi} \Gamma^\mu \psi = -\bar{\psi} \Gamma^\mu \psi$

sum for $\Theta =$

$$s(\overline{\sigma}^{\text{f}}) = \overline{\sigma}^{\text{f}} + \overline{\sigma}^{\text{f}}_2(s) = 2(\overline{\sigma})^{\text{f}}_2$$

$$\mathcal{L}_S = -m \left(\dot{\alpha} \frac{1}{2} (-\Pi_\alpha \Pi_\alpha)^{1/2} + 2(-\bar{\Theta} \Gamma^t \delta \Theta) (\dot{\chi}_\mu - \bar{\Theta} \Gamma_\mu \dot{\Theta}) \right) \Rightarrow p \cdot \Gamma \dot{\Theta} = 0$$

$$(p_r r^r)(p_r r^r) \dot{\theta} = \frac{1}{2} (p_r p_r r^r r^r - p_r p_r r^r r^r) \dot{\theta} = \frac{1}{2} p_r p_r (\underbrace{r^r r^r + r^r r^r}_{2r^r}) \dot{\theta} = p^r \dot{\theta} = -m^r \dot{\theta} = 0 \quad \text{for } m \neq 0 \Rightarrow \dot{\theta} = 0$$

no fermion dynamics

Spec. eqn are $(\mathbf{P} \cdot \mathbf{r} + m \mathbf{r}_n) \dot{\theta} = 0$, then $(\mathbf{P} \cdot \mathbf{r} + m \mathbf{r}_n)^2 = (\mathbf{P} \cdot \mathbf{r})^2 + \mathbf{P} \cdot \mathbf{r} m \mathbf{r}_n + m \mathbf{r}_n \mathbf{P} \cdot \mathbf{r} + m^2 \mathbf{r}_n^2 = \mathbf{P}^2 + \mathbf{r}_n^2 = 0$

Can achieve this by $S_2 = -m \int dt \bar{\theta} \Gamma_4 \dot{\theta}$

$$S = -m \int d\tau \sqrt{-\Pi_0 \cdot \Pi_0} - m \int d\tau \bar{\Theta} \Gamma_0 \dot{\Theta} \quad \text{let } \bar{\tau} = \bar{\tau}(\tau), \quad d\bar{\tau} = \frac{d\bar{\tau}}{d\tau} d\tau,$$

$$= \int dt \frac{dz}{dz} \left(-\left(\frac{dz}{dt}\right) \left(\frac{dz}{dt}\right)^r - \partial_t \left(\frac{dz}{dt}\right) \right) (-) \checkmark$$

- 5.2 In order to obtain a non-trivial massless limit of eq. (5.21), it is useful to first restore the auxiliary field $\epsilon(x)$:
- (i) Re-express the massive D0 brane action using $\epsilon(x)$.
 - (ii) Find the massless limit of the D0 brane action.
 - (iii) Verify the κ symmetry of the massless D0 brane action.

For normal particle $S = -m \int dt \sqrt{-\dot{x} \cdot \dot{x}}$, $\tilde{S} = \frac{1}{2} \int dt (e^2 \dot{x}^2 - m^2 e)$

Note, second term above does not need to be redefined, because for $m=0$ $(P \cdot \Gamma + m \Gamma_0) \dot{\Theta} = P \cdot \Gamma \dot{\Theta}$ which you can just get from the first term.

Consider,

$$\tilde{S} = \frac{1}{2} \int dt (e^2 \Pi_0 \cdot \Pi_0 - m^2 e) - m \int dt \bar{\Theta} \Gamma_0 \dot{\Theta}$$

$$\delta \tilde{S} = \frac{1}{2} \int dt (-e^2 \Pi_0 \cdot \Pi_0 \delta e - m^2 \delta e) \Rightarrow m^2 e + \Pi_0 \cdot \Pi_0 = 0 \Rightarrow e = \frac{1}{m^2} \sqrt{-\Pi_0 \cdot \Pi_0}$$

$$\tilde{S} = \frac{1}{2} \int dt \left(-\frac{m^2 (-\Pi_0 \cdot \Pi_0)}{\sqrt{-\Pi_0 \cdot \Pi_0}} - m \sqrt{-\Pi_0 \cdot \Pi_0} \right) = S$$

As $m \rightarrow 0$, $\tilde{S} \rightarrow \frac{1}{2} \int dt e^2 \Pi_0 \cdot \Pi_0$

$$\delta \Pi_0 = -2 \delta \bar{\Theta} \Gamma^0 \dot{\Theta} = -2 \bar{E} P \cdot \Gamma^0 \dot{\Theta}$$

$$\delta S_{m=0} = \frac{1}{2} \int dt \left(-e^2 \Pi_0 \cdot \Pi_0 \delta e + \frac{2e^2 \Pi_0 \cdot \delta \Pi_0}{-4e^2 \Pi_0 \cdot \bar{E} P \cdot \Gamma^0 \dot{\Theta}} \right)$$

5.3 Prove that for a pair of Majorana spinors $\Theta, \bar{\Theta}_2$: $\bar{\Theta}_1 \Gamma_{\mu_1 \dots \mu_n} \Theta_2 = (-1)^{n(n+1)/2} \bar{\Theta}_2 \Gamma_{\mu_1 \dots \mu_n} \Theta_1$

Dirac matrices: $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}$, Majorana condition $\bar{\Theta} = \bar{\Theta}^C$, Charge conjugation $C \Gamma_\mu C^{-1} = -(\Gamma_\mu)^T$
 (Lorentz: $\Gamma_{5+1} \psi = 2\psi$ (Chirality)
 Majorana: $\psi = C \bar{\psi}^T$ (Real)
 $C \Gamma_\mu C^{-1} = \frac{\Gamma_\mu C}{-1} = \frac{\Gamma_\mu^T}{-1} = -\Gamma_\mu^T$
 $C \Gamma_\mu C^{-1} = \Gamma_\mu C (-C) = -\Gamma_\mu C C^{-1} = -\Gamma_\mu$
 $\Rightarrow \Gamma_\mu \in \mathbb{R}$
 $= -\Gamma_\mu^T = -\Gamma_\mu^T \Rightarrow \Gamma_\mu^T = \Gamma_\mu = \Gamma_\mu^T$

Ex. 5.1: $\bar{\Theta}_1 \Gamma_\mu \Theta_2 = \bar{\Theta}_1^T \Gamma_\mu^T \Theta_2 = -\bar{\Theta}_2^T \Gamma_\mu^T \Theta_1 = \bar{\Theta}_2^T \Gamma_\mu^T \Theta_1 = \left| \begin{matrix} \mu=0: -\bar{\Theta}_2^T \Gamma_0 \Theta_1 = -\bar{\Theta}_2 \Gamma_0 \Theta_1 \\ \mu=i: \bar{\Theta}_2^T \Gamma_i \Theta_1 = -\bar{\Theta}_2 \Gamma_i \Theta_1 \end{matrix} \right|$

$$\bar{\Theta}_1 \Gamma_{\mu_1 \dots \mu_n} \Theta_2 = (\bar{\Theta}_1^T \Gamma_{\mu_1 \dots \mu_n}^T \Theta_2^T)^T = -\bar{\Theta}_2^T \Gamma_{\mu_1 \dots \mu_n}^T \Theta_1^T = \bar{\Theta}_2^T \Gamma_{\mu_1 \dots \mu_n}^T \Theta_1$$

$$2 \times (1 \cdot 2 + \dots + 100) = \frac{1+2+\dots+100}{100+99+\dots+1} = \frac{101 \times 100}{2}$$

If $\mu_i = i$, $(-1)^n \Gamma_{\mu_1 \dots \mu_n} \Gamma_{\mu_1 \dots \mu_n} = (-1)^n \Gamma_{\mu_1 \dots \mu_n} \Gamma_{\mu_1 \dots \mu_n} = (-1)^n (n-1)(n-2) \dots (n-(n-1)) = 1 \cdot 2 \cdot \dots \cdot n-1 + n = \sum_{i=1}^n i = \frac{n(n+1)}{2}$

If $2 \mu_i$ are the same $\mu_1 \mu_2 \dots \mu_{n-1} \mu_n \Rightarrow$ commute twice $\Rightarrow (-1)^{n(n+1)/2 - 2} = (-1)^{n(n+1)/2}$

If some $\mu_i = 0$, $\mu_1 \mu_2 \dots \mu_{n-1} \mu_n$

5.4 Derive the relevant Fierz identities for Majorana-Weyl spinors in 10d and use them to prove that $\Gamma^I d\Theta d\bar{\Theta} \Gamma_I d\Theta = 0$.

10d:  $[0010] \dim = 2^{5-1} = 16 \rightarrow$ For H.O. the basis for $\mathbf{16}$ has $\frac{16 \cdot 16}{256}$ elements
 $[00010][00001] = \frac{[00011]}{45} + \frac{[01000]}{45} + \frac{[00000]}{1}$
 $\frac{1}{45} [00000] \quad \frac{1}{45} [01000]$
 $\left(\frac{10}{4}\right) = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 3 \quad \left(\frac{10}{2}\right) = \frac{10 \cdot 9}{2} = 45$
 $= 210$
 $[00010] = 1 \quad [00001] = 3 \Rightarrow [00011] = 0$

$[00010][00010] = \frac{[00020]}{10} + \frac{[10000]}{10} + \frac{[00100]}{1}$
 $\frac{1}{10} [00000] \quad \frac{1}{10} [10000]$
 $\frac{1}{2} \left(\frac{10}{4}\right) = \frac{1}{2} \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 4 \quad \left(\frac{10}{2}\right) = \frac{10 \cdot 9}{2} = 45$
 $= 2 \cdot 9 = 18$
 $= 2 \cdot 9 = 18$

Fierz transformation identity: $\chi \bar{\psi} = \frac{1}{2^5} \sum_{p=0}^5 \frac{1}{p!} \gamma^{\mu_1 \dots \mu_p} \bar{\psi} \gamma_{\mu_1 \dots \mu_p} \chi$

5.5 Verify that the action $S_2 = \frac{1}{2} \int d\sigma \varepsilon^{\alpha\beta} \Omega_{\alpha\beta}$ with $\Omega_2 = c(\bar{\Theta} \Gamma_\mu d\Theta - \bar{\Theta}^* \Gamma_\mu d\Theta^*) dX^\mu - c \bar{\Theta}^* \Gamma_\mu d\Theta \bar{\Theta}^* \Gamma^\mu d\Theta$ is invariant under susy transformations.

PSIDE:

In type IIA: $\Gamma_\mu \Theta^a = (-\gamma^\mu)^{ab} \Theta^b$ $\neq$ In type IIB: $\Gamma_\mu \Theta^a = \Theta^a$ w/ $A=1,2$

The natural generalisation of the Nambu-Goto action is

$S_{\text{NG}} = -\frac{1}{2\pi} \int d\sigma \sqrt{-\det(\partial_\mu \bar{X}^\alpha \partial_\nu X_\alpha)}$ $\rightarrow S_2 = -\frac{1}{4\pi} \int d\sigma \sqrt{-\Theta^* \Theta} = -\frac{1}{4\pi} \int d\sigma \sqrt{-\det(\Pi_\mu - \Pi_\mu^*)}$
 $\rightarrow \Pi_\mu^* = \partial_\mu \bar{X}^a - \bar{\Theta}^* \Gamma^\mu \partial_\mu \Theta^a$

L symmetry: $\delta X^\mu = \bar{\Theta}^* \Gamma^\mu \delta \Theta^a = -\bar{\Theta}^* \Gamma^\mu \Theta^a \Rightarrow \delta \Pi_\mu^* = -2 \bar{\Theta}^* \Gamma^\mu \partial_\mu \Theta^a \Rightarrow \delta S_2 = \frac{3}{8\pi} \int d\sigma \sqrt{-\Theta^* \Theta} \Pi_\mu^* \delta \bar{\Theta}^* \Gamma_\mu \partial_\mu \Theta^a$

For a p-brane, S_2 naturally has the form of an integral of a (p+1)-form, here:

$S_2 = \int \Omega_2 = \frac{1}{2} \int d\sigma \varepsilon^{\alpha\beta} \Omega_{\alpha\beta}$
 $\rightarrow$ manifestly diffeomorphism invariant

To make the symmetries manifest, we formally introduce an additional dimension by considering $\Omega_3 = d\Omega_2$. If M is the string worldsheet

$\int_0^1 d\sigma \int_0^1 d\tau \Omega_2 = \int_{\partial M} \Omega_2$
 $\rightarrow$ not invariant under susy, but transforms as total derivative

There are 3 one-forms that are susy invariant: $d\Theta$, $d\Theta^*$, and $\Pi^I = dX^I - \bar{\Theta}^* \Gamma^I d\Theta^a \rightarrow \Omega_3$ should be a Lorentz inv. combination:

$\Omega_3 = c(d\bar{\Theta}^* \Gamma_\mu d\Theta - d\bar{\Theta}^* \Gamma_\mu d\Theta^*) \Pi^\mu$
 $\rightarrow$ need $d\Omega_3 = 0$

$$\delta \theta^{\alpha} = \xi^{\alpha}, \quad \delta W = 2^{\alpha} \Gamma^{\alpha} \theta^{\alpha}$$

$$\delta S_1 = \delta \left(\int_{\mathcal{M}} \mathcal{L}_1 \right) = \delta \left(\int_{\mathcal{M}} \mathcal{L}_1 \right) = \int_{\mathcal{M}} \delta \mathcal{L}_1 = 0$$

invariant by construction.

5.6) Prove the identity $\{\Gamma_{\mu\nu}, \Gamma_{\rho\sigma}\} = -2\eta_{\mu\rho}\eta_{\nu\sigma} + 2\eta_{\mu\sigma}\eta_{\nu\rho} + 2\Gamma_{\mu\rho\sigma}$.

$$\Gamma_{\mu\nu} = \frac{1}{2}(\Gamma_{\mu\nu} - \Gamma_{\nu\mu}) = \Gamma_{\mu\nu} - \eta_{\mu\nu}$$

$$\eta_{\mu\nu}(\mathcal{L}_1, \rho\sigma) = 2\eta_{\mu\rho}\rho\sigma$$

$$\{\Gamma_{\mu\nu}, \Gamma_{\rho\sigma}\} = \{\mu\nu - \eta_{\mu\nu}, \rho\sigma - \eta_{\rho\sigma}\} = \{\mu\nu, \rho\sigma\} - \{\eta_{\mu\nu}, \rho\sigma\} - \{\mu\nu, \eta_{\rho\sigma}\} + \{\eta_{\mu\nu}, \eta_{\rho\sigma}\}$$

$$= \{\mu\nu, \rho\sigma\} - 2\eta_{\mu\rho}\rho\sigma - 2\eta_{\mu\sigma}\rho\nu + 2\eta_{\mu\nu}\rho\sigma - 2\Gamma_{\mu\rho\sigma} + 2\Gamma_{\mu\nu\sigma}$$

$$= \mu\rho\sigma - \rho\sigma\mu - 2\eta_{\mu\rho}\rho\sigma - 2\eta_{\mu\sigma}\rho\nu + 2\eta_{\mu\nu}\rho\sigma + 2\Gamma_{\mu\rho\sigma} - \frac{1}{4\pi^2} \left(\mu\rho\sigma + \mu\rho\sigma + \mu\rho\sigma - \mu\rho\sigma - \mu\rho\sigma - \mu\rho\sigma \right.$$

$$\left. - \nu\rho\sigma - \nu\rho\sigma - \nu\rho\sigma + \nu\rho\sigma + \nu\rho\sigma + \nu\rho\sigma \right.$$

$$\left. + \rho\sigma\mu + \rho\sigma\mu + \rho\sigma\mu - \rho\sigma\mu - \rho\sigma\mu - \rho\sigma\mu \right.$$

$$\left. - \sigma\mu\rho - \sigma\mu\rho - \sigma\mu\rho + \sigma\mu\rho + \sigma\mu\rho + \sigma\mu\rho \right)$$

$$= -\frac{1}{4\pi^2} \left(4\mu\rho\sigma - 4\rho\mu\sigma - 4\mu\rho\sigma + 4\rho\mu\sigma \right.$$

$$\left. + 4\rho\sigma\mu - 4\rho\sigma\mu - 4\rho\sigma\mu + 4\rho\sigma\mu \right.$$

$$\left. + 4\sigma\mu\rho - 4\sigma\mu\rho - 4\sigma\mu\rho + 4\sigma\mu\rho \right.$$

$$\left. + 4\nu\rho\sigma - 4\nu\rho\sigma - 4\nu\rho\sigma + 4\nu\rho\sigma \right.$$

$$\left. + 4\nu\sigma\rho - 4\nu\sigma\rho - 4\nu\sigma\rho + 4\nu\sigma\rho \right.$$

$$\left. + 4\rho\sigma\mu - 4\rho\sigma\mu - 4\rho\sigma\mu + 4\rho\sigma\mu \right)$$

$$\mu\rho\sigma = -\rho\mu\sigma + 2\rho\sigma\mu = \mu\rho\sigma - 2\rho\sigma\mu + 2\rho\sigma\mu$$

5.9 (i) Show that $\text{tr}(F \wedge F)$ is closed and gauge invariant.

(ii) Show that for the local exp. $\text{tr}(F \wedge F) = d\omega_3$, $\omega_3 = \text{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$.

(iii) Similarly $\text{tr} F^2 = d\omega_2$. Find ω_2 .

ASIDE: Gauge Anomalies and their cancellation.

When a symmetry of a classical theory is broken by radiative corrections, such that there is no choice of local counter terms,

the symmetry is anomalous.

→ anomalies that break a global symmetry: good

→ " " local gauge symmetry: bad

Def. A Lie-algebra valued p -form A , locally looks like $A = A^a \otimes X^a = \frac{1}{p!} A^a_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} X^a$ in some rep.
↑ Lie algebra elem.

For Yang-Mills gauge fields: $A = A^a_{\mu}(x) dx^\mu X^a$
↑ 1-form part

The field strength is the commutator of covariant derivatives $D_\mu = \partial_\mu + A_\mu$:

$$\begin{aligned} F_{\mu\nu} \psi &= [D_\mu, D_\nu] \psi \\ &= (\partial_\mu + A_\mu)(\partial_\nu \psi + A_\nu \psi) - (\nu \leftrightarrow \mu) \\ &= (\cancel{\partial_\mu \partial_\nu} \psi + \cancel{\partial_\mu A_\nu} \psi + A_\nu \cancel{\partial_\mu \psi} + A_\mu \cancel{\partial_\nu \psi} + A_\mu A_\nu \psi) - (\nu \leftrightarrow \mu) \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu) \psi + (A_\mu A_\nu - A_\nu A_\mu) \psi \\ \Rightarrow F_{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu] \\ dA &= d(A_\mu dx^\mu) = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \\ A \wedge A &= (A_\mu dx^\mu) \wedge (A_\nu dx^\nu) = A_\mu A_\nu (dx^\mu \wedge dx^\nu) = \frac{1}{2} (A_\mu A_\nu - A_\nu A_\mu) dx^\mu \wedge dx^\nu \\ \Rightarrow F &= dA + A \wedge A \end{aligned}$$

A gauge transformation is a local transformation by a matrix $g(x) \in G$: $\psi'(x) = g(x)\psi(x)$

We want D_μ to transform in a similar manner.

$$D'_\mu \psi' = g(D_\mu \psi)$$

$$(d + A')(g\psi) = g(d + A)\psi$$

$$(dg)\psi + g(d\psi) + A'g\psi = g(d\psi) + gA\psi$$

$$\begin{aligned} \Rightarrow dg + A'g &= gA \quad \Rightarrow A' = gAg^{-1} - (dg)g^{-1} \quad \begin{matrix} g = \mathbb{1} + \lambda \\ g^{-1} = \mathbb{1} - \lambda \end{matrix} \\ &= (\mathbb{1} + \lambda)A(\mathbb{1} - \lambda) - (d(\mathbb{1} + \lambda))(\mathbb{1} - \lambda) \\ &= A + \lambda A - A\lambda - d\lambda = A - [A, \lambda] - d\lambda = A - \mathbb{S}_\lambda A \end{aligned}$$

$$\psi' = g^{-1}\psi, \quad D'_\mu \psi' = g^{-1}(D_\mu \psi) = D'_\mu(g^{-1}\psi) \Rightarrow g^{-1}D = D'g^{-1} \Rightarrow D = g^{-1}D'g \Rightarrow F' = (D')^2 = g^{-1}Fg = (\mathbb{1} - \lambda)F(\mathbb{1} + \lambda) = F - \lambda F + F\lambda = F - [\lambda, F] - \mathbb{S}_\lambda F$$

Gravity is described in an almost identical manner. The spin-connection one-form

$$\omega = \omega_{\mu}^a(x) \Lambda^a dx^\mu$$

↑ func. rep of Lorentz alg.

↓ gauge field for local Lorentz's sym.

The curvature 2-form is $R = d\omega + \omega \wedge \omega$

Under an infinitesimal local Lorentz transformation: $S_0 \omega = d\Theta + [\omega, \Theta]$ and $S_0 R = [R, \Theta]$

Characteristic classes are special forms that allow ω to calculate topological invariants (I.M.s) by integrating local field strengths.

(1) $dX(R, F) = 0$ ↪ let $A_\mu = (-b)A_\mu + tA_\mu$, $t \in [0, 1]$
consider $dX = X(\partial_\mu) dx^\mu$ then
 $\int_{\partial M} dX = 0 = \int_{\partial M} X = \int_{\partial M} X(A_\mu) = \int_{t=0}^1 X(A_\mu) = Q(A_0) = Q(A_1)$

(2) $S_0 X(R, F) = S_0 Q(R, F) = 0$

e.g.: $\text{tr}(F \wedge \dots \wedge F) \neq \text{tr}(R \wedge \dots \wedge R)$ & Polynomials of these.

Bianchi: $DF = dF + [A, F] = 0 \Rightarrow dF = F \wedge A - A \wedge F$

(i) $d(\text{tr} F \wedge F) = \text{tr}(dF) \wedge F + \text{tr} F \wedge (dF) = 2 \text{tr} F \wedge dF = \text{tr}(F \wedge F \wedge A - F \wedge A \wedge F) = 0$

$(\text{tr} F \wedge F)' = \text{tr} F' \wedge F' = \text{tr} g^{-1} F g \cdot g^{-1} F g = \text{tr} g g^{-1} F F = \text{tr} F \wedge F$

(ii) $\text{tr}(F \wedge F) \propto C_2$. since $d(\text{tr} F \wedge F) = 0 \Rightarrow \text{tr}(F \wedge F) = d\omega_3$

$d\omega_3 = d\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = dA \wedge dA - A \wedge d^2 A + \frac{2}{3} (dA \wedge A \wedge A - A \wedge dA \wedge A + A \wedge A \wedge dA)$

$\text{tr}(F \wedge F) = \text{tr}\left((dA + A \wedge A) \wedge (dA + A \wedge A)\right) = \text{tr}\left(dA \wedge dA + \underbrace{dA \wedge A \wedge A + A \wedge A \wedge dA}_{2 dA \wedge A \wedge A} + A \wedge A \wedge A \wedge A\right)$

(iii) $\text{tr} F' = \text{tr}\left((dA + A \wedge A)^2\right) = \text{tr}\left((dA)^2 + 4 dA \wedge A^2 + 6 (dA)^2 \wedge A^2 + A^4\right)$

$\omega_3 \supset b(A^4): a(dA \wedge \dots \wedge A \wedge dA) = 7a dA \wedge A^4 \Rightarrow a = \frac{4}{3}$

$\omega_3 \supset \text{tr}(b A^2 dA): b 5 A^2 (dA)^2 - b A^2 d^2 A \Rightarrow b = \frac{6}{5}$

$\omega_3 \supset \text{tr}(c A^3 dA^2): c 3 A^2 (dA)^3 \Rightarrow c = \frac{4}{3}$

$\omega_3 \supset \text{tr}(e A^4 dA): e (dA)^4 \Rightarrow e = 1$

(5.10) Check the identity: $\text{Tr} e^{iF} = \frac{1}{2} (\text{tr} e^{iF})^2 - \frac{1}{2} \text{tr} e^{2iF} = \frac{1}{2} (\text{tr} \cos F)^2 - \frac{1}{2} \text{tr} \cos 2F$, for $SO(N)$.

ASIDE:

partition function picks up a phase factor
anomalous symmetry: $Z[A] = e^{i\alpha(g, A)} Z[A]$

Characterisation of Anomalies:

not really living in $D=2n$.

YM & Lorentz anomalies in $D=2n$ are encoded in a characteristic class that is a $(2n+2)$ -form, I_{2n+2} .

The physical anomaly is characterised by a $2n$ -form Θ_{2n} as

$\delta S_{\text{eff}} = \int \Theta_{2n}$ w/ $I_{2n+2} = d\omega_{2n+1}$ and $\delta \omega_{2n+1} = d\Theta_{2n}$

The total anomaly is the sum of contributions from each chiral field $I_{2n+2} = \sum_k I_k^{(n)}$

Every possible anomaly contribution $I_{2n+2}^{(G)}$ (due to Alvarez-Gaume & Witten)

(a) Left-handed Weyl fermion in ρ representation of YM gauge group:

Dirac-Bogomol'nyi Chern character

$$I_{2n}(R, F) = \left[\hat{A}(R) \text{tr}_\rho e^{iF} \right]_{2n+2}$$
[...] 2n+2: select 2n+2 form

where $\hat{A}(R) = \prod_{i=1}^n \frac{\lambda_i/2}{\sinh(\lambda_i/2)}$ w/ λ_i are the eigenvalue two-forms of the curvature: $R = \begin{pmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ & & & 0 & \lambda_n \\ & & & -\lambda_n & 0 \end{pmatrix}$

$$= 1 + \frac{1}{48} \text{tr} R^2 + \frac{1}{16} \left(\frac{1}{288} (\text{tr} R^2)^2 + \frac{1}{360} \text{tr} R^4 \right) + \dots$$

(b) Left-handed Weyl gravitino, always a singlet of any YM group:

$$I_{32}(R) = \left(\sum_j 2 \cosh \lambda_j - 1 \right) \prod_i \frac{\lambda_i/2}{\sinh(\lambda_i/2)}$$

(c) Self-dual tensor

Hirzebruch L-function

$$I_8(R) = -\frac{1}{8} L(R) \quad \text{w/} \quad L(R) = \prod_{i=1}^4 \frac{\lambda_i}{\cosh(\lambda_i)}$$

The fields of opposite chirality have the same terms but w/ opposite sign.

A useful identity is:

$$\hat{A}(R/2) = \sqrt{L(R/4) \hat{A}(R)}$$

$$([2000] + [0000])^2 = [2000] + [0100] + [0000] + 2[1001] + 2[0010] + [0002] + [0100] + [0000]$$

In type IIB, there are 2 left-handed Majorana-Weyl gravitinos & 2 right-handed Majorana-Weyl spinors and a s.d. boson.

The total anomaly is given by the 12-form part of: $I(R) = I_{32}(R) - I_{12}(R) + I_8(R) = 0$ (A.-G. & W.)

$[0000]_A$: NOT CHIRAL | $[0000]_A$: CHIRAL

YM

$$([2000] + [0000]) \times A \cdot [0000] = VA + G$$

In type I, we have $([0000] + [0000]) [0000] = [2000] + [0100] + [0000] + [1001] + [0010]$, so the anomaly is

$$\frac{1}{2} (I_{32}(R) - I_{12}(R))_{12} = -\frac{1}{2} (I_8(R))_{12} = \frac{1}{16} (L(R))_{12} \quad \text{w/o gauge contribution}$$

$$\frac{1}{16} (L(R))_{12} = \frac{1}{2} [\hat{A}(R) \text{tr}_{\text{adj}} e^{iF}]_{12} \quad \text{Tr} \pm \text{tr}_{\text{adj}} \quad \text{tr} = \text{tr}_{\text{fund.}}$$

Chern-character factorization property: $\text{tr}_{\rho_1 \otimes \rho_2} e^{iF} = (\text{tr}_{\rho_1} e^{iF}) (\text{tr}_{\rho_2} e^{iF})$

$$[0000]^2 = [2000] + [0100] + [0000]$$

$$(\text{tr} e^{iF})^2 = \text{tr}_{[0000], [0000]} e^{iF} + \text{Tr} e^{iF}$$

Sym²[0000]

$$\text{tr}_{\text{Sym}^2 A} e^{iF} = \frac{1}{2} (\text{tr}_A e^{iF})^2 + \text{tr}_A e^{2iF}$$

$$\text{Tr} e^{iF} = \frac{1}{2} (\text{tr} e^{iF})^2 - \frac{1}{2} \text{tr} e^{2iF}$$

$$\chi(g)^2 = \chi_{\text{Sym}^2}(g) + \chi_A(g)$$

$$\chi(g^2) = \chi_{\text{Sym}^2}(g^2) - \chi_A(g^2)$$

adj rep.

$$\langle i, j | \tau(g) \otimes g | i, j \rangle = \langle j | i \otimes g | j | i \rangle = g_{ij} g_{ji} = \text{tr}(g^2)$$

3.11 (i) Using $\Psi = \frac{1}{2} \sqrt{\hat{A}(R)} \cos F - 16 \sqrt{L(RH)}$ obtain an expression for Ψ_4 .

(ii) Apply the descent formalism to obtain a formula for Θ_2 in $\Theta_0 = 2\Theta_2\Psi_2$

Ans:

The type I anomaly can now be written as

$$I = \left[\frac{1}{2} \hat{A}(R) \text{Tr} e^{\hat{F}} + \frac{1}{16} L(R) \right]_{12} = \frac{1}{2} (\text{tr} \cos F)^2 - \frac{1}{4} \text{tr} \cos(2F) \quad \text{L of 6th order in R's and F's}$$

$F^2 = -F \Rightarrow \text{tr}(F^{\text{even}}) = 0$

For the 12th order term: $I = I' = \frac{1}{4} \hat{A}(R) (\text{tr} \cos F)^2 - \frac{1}{4} \left(\frac{2^6}{16} \right) \hat{A}(R/2) \text{tr} \cos F + \frac{1}{16} (4^6) L(RH)$

2, 4, 8, 16, 32, 64 4, 16, 64, 256, 1024, 4096

Which we can factorise.

$$I' = \left(\frac{1}{2} \sqrt{\hat{A}(R)} \text{tr} \cos F - 16 \sqrt{L(RH)} \right)^2 = \frac{1}{4} \hat{A}(R) (\text{tr} \cos F)^2 - 16 \sqrt{\hat{A}(R) L(RH)} \text{tr} \cos F + 256 L(RH)$$

$= \hat{A}(R/2)$

There is no choice of N st. $[I]_{12} = [I']_{12} = 0$, but one can introduce a counter-term that cancels the anomaly I'

$[I]_{12}$ factorises into the product of a 4-form and an 8-form.

A priori, Ψ is a sum $\Psi_0 + \Psi_4 + \Psi_8 + \dots$, but if Ψ_0 vanishes we instead have $[I]_{12} = [(16, \dots)]_{12} = 256 \Psi_8$

$$\Psi_8 = \frac{1}{2} \sqrt{\hat{A}(R)} \text{tr} \cos F - 16 \sqrt{L(RH)} = \frac{N-32}{2} \Rightarrow N=32 \Rightarrow 30(32)$$

We can write $\Psi = \underbrace{\left(\frac{1}{2} \sqrt{\hat{A}(R)} \text{tr} \cos F \right)}_{\Psi_B} + \underbrace{\left(-16 \sqrt{L(RH)} \right)}_{\Psi_C}$

Ψ_B boundary of string worldsheet (D-brane) Ψ_C cross-cap (orientifold plane)

Then the anomaly contribs. from distinct topologies are

$$I' = \Psi^2 = \Psi_B^2 + 2\Psi_B\Psi_C + \Psi_C^2$$

(i) $\Psi = \frac{1}{2} \sqrt{\hat{A}(R)} \text{tr} \cos F - 16 \sqrt{L(RH)} = \frac{1}{2 \sqrt{\hat{A}(R)}} \left(\hat{A}(R) \text{tr} \cos F - 32 \hat{A}(R/2) \right)$

$R^2 \in \mathbb{Z}$

$\hat{A}(R) = 1 + \frac{1}{48} \text{tr} R^2 + \dots$ $\hat{A}(R/2) = 1 + \frac{1}{192} \text{tr} R^2 + \dots$ $\text{tr} \cos F = \text{tr} \left(1 - \frac{1}{24} F^2 + \dots \right) = N - \frac{1}{2} \text{tr} F^2$ $(\hat{A}(R))^{1/2} = 1 - \frac{1}{96} \text{tr} R^2 + \dots$

$\Rightarrow \Psi = \frac{1}{2} \left(1 - \frac{1}{96} \text{tr} R^2 + \dots \right) \left\{ \left(1 + \frac{1}{48} \text{tr} R^2 + \dots \right) \left(N - \frac{1}{2} \text{tr} F^2 \right) - 32 \left(1 + \frac{1}{192} \text{tr} R^2 + \dots \right) \right\}$

$= N - \frac{1}{2} \text{tr} F^2 + \frac{N}{96} \text{tr} R^2$

$N=32$

$= \frac{1}{2} \left(1 - \frac{1}{96} \text{tr} R^2 + \dots \right) \left(\frac{(N-32)}{2} - \frac{1}{2} \text{tr} F^2 + \frac{32}{96} \text{tr} R^2 - \frac{1}{6} \text{tr} R^2 \right) + \dots = \frac{1}{4} (\text{tr} R^2 - \text{tr} F^2)$

$= \frac{1}{2} \text{tr} R^2$

(ii)

The anomaly polynomial $I_4 \in \mathcal{I}^{4d-2}$ satisfies $dI_4 = 0 \neq \delta I_4 = 0$.

→ has to vanish/cancel for theory to be consistent.

" anomaly variation $G_4 \in \mathcal{I}^{4d}$ descends from this as $I_4 = dI_3 \neq \delta I_4 = dG_4$.

$$d\text{tr} R^2 = 0 = d\text{tr} F^2$$

Chern-Simons 3-form

Since $\zeta_4 = \frac{1}{4}(\text{tr} R^2 - \text{tr} F^2)$, $d\zeta_4 = 0$ and locally we have $\zeta_4 = d\omega_3$

$$\Rightarrow \omega_3 = \frac{1}{4}(\omega_{3L} - \omega_{3F}) \quad \text{st. } d\omega_{3L} = \text{tr} R^2 \neq d\omega_{3F} = \text{tr} F^2$$

Using the Leibniz rule in reverse: $2\zeta_4 \zeta_8 = 2(d\omega_3) \zeta_8 = 2d(\omega_3 \zeta_8) - 2\omega_3 \wedge d\zeta_8 = [I]_4 = dI_3 \Rightarrow I_3 = 2\omega_3 \zeta_8$

Now, using the fact that $dG_4 = \delta\omega_3 = \delta\left(\frac{1}{4}(\omega_{3L} - \omega_{3F})\right) = \frac{1}{4}(\delta\omega_{3L} - \delta\omega_{3F})$

$$d\omega_{3F} = \text{tr} F^2 = \text{tr}(dA \wedge A + A \wedge dA + A \wedge A \wedge A) = \text{tr}(dA \wedge dA + dA \wedge A + A \wedge dA + A^3) = \text{tr}(dA \wedge dA + 2dA \wedge A + A^3)$$

$$\Rightarrow \omega_{3F} = \text{tr}\left(A \wedge dA + \frac{2}{3} A^3\right)$$

Under an infinitesimal gauge transformation $\delta A = D\Lambda = d\Lambda + [A, \Lambda]$, so

$$\begin{aligned} \delta\omega_{3F} &= \text{tr}\left(\delta A \wedge dA + A \wedge d(\delta A) + 2(\delta A) \wedge A^2\right) \\ &= \text{tr}\left(\delta A \wedge dA + d(A \wedge \delta A) - dA \wedge \delta A + 2\delta A \wedge A^2\right) \\ &= 2\text{tr} \delta A \wedge F + d\text{tr} \delta A \wedge A \\ &= 2\text{tr}\left(d\Lambda \wedge F + \underbrace{A \wedge \Lambda \wedge F - \Lambda \wedge A \wedge F}_{N[F, A] = \Lambda dF} + d\text{tr}\left(d\Lambda \wedge A + \underbrace{A \wedge \Lambda \wedge A - \Lambda \wedge A^2}_{D\Lambda = 0}\right)\right) \\ &= 2d\text{tr} \Lambda F + d\text{tr} d\Lambda \wedge A \\ &= d\text{tr}\left(\underbrace{2\Lambda F + d\Lambda \wedge A}_{G_{2F}}\right) \end{aligned}$$

 Bouchard.

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