String Theory

Based on lectures by Prof. Amihay Hanany
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Abstract

These notes are based on the string theory lecture course by Prof. Amihay Hanany, taught at Imperial College London in 2025. While they broadly follow the structure of the course, at times I decided to reorganise and/or add additional material. There are very likely many mistakes throughout this work, from typos to factually wrong statements.

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We will briefly review some of the relevant aspects of representation theory. It is assumed that the reader is familiar with how highest weights work and that they correspond to different representations of an algebra.

2.1 Basic tools

Let us begin by introducing the **Dynkin labels**. Recall that a highest weight μ can be written as a linear combination of fundamental weights λ

$$\mu = \sum_{i=1}^{r} n_i \lambda_i \,, \tag{1}$$

where r is the rank of the algebra and $n_i \in \mathbb{Z}^+$. The Dynkin labels are simply the coefficients n_i written as a vector

$$[n_1 n_2 \cdots n_r] \,. \tag{2}$$

We are often interested in performing various operations on different representations. Let us start with the **tensor product**, and suppose we have the representations [10] and [01] of A_2. There are now multiple ways to tackle the tensor product, either one does the actual multiplication of weights (or uses characters to encode them), or one makes use of general rules[1]:

- the dimensions have to (obviously) be the same on both sides of the equation,
- the charge under the centre of the group is conserved,
- in general $[n_1 ... n_r][m_1 ... m_r] = [(n_1 + m_1), (n_2 + m_2), ..., (n_r + m_r)] + ...$.

Let us begin by multiplying the weights, as schematically shown in figure 1. This method is great for 2-dimensional root systems, but gets more difficult to visualise in higher dimensions. In terms of characters, we note that

$$[10] = x + \frac{y}{x} + \frac{1}{y} \tag{3}$$

$$[01] = y + \frac{x}{y} + \frac{1}{x} \tag{4}$$

$$[11] = xy + \frac{x^2}{y} + \frac{x}{y^2} + \frac{1}{xy} + \frac{y}{x^2} + \frac{y^2}{x} + 2 \,, \tag{5}$$

$$[00] = 1 \,, \tag{6}$$

[1] This is not an exhaustive list to any extent. For example, one can also use Young tableaux for tensor products of $SU(n)$ representations.

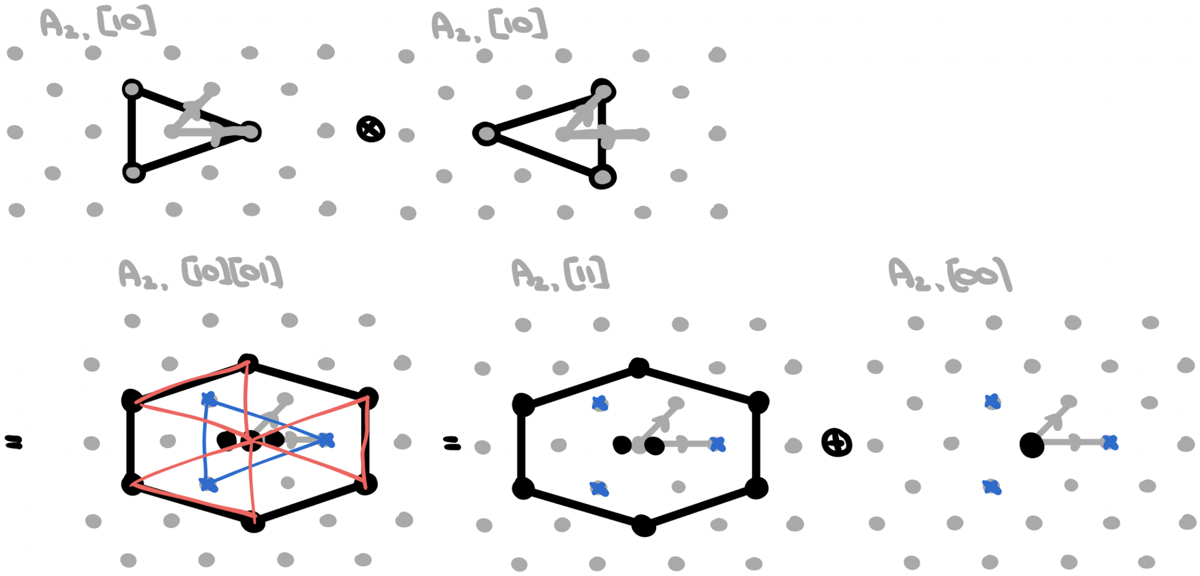


Figure 1: Weight computation for [10][[01] = [11] + [00].

and the tensor product is simply the product of the characters

$$[10][01] = (x + \frac{y}{x} + \frac{1}{y})(y + \frac{x}{y} + \frac{1}{x}) \tag{7}$$

$$= xy + \frac{x^2}{y} + 1 + \frac{y^2}{x} + 1 + \frac{y}{x^2} + 1 + \frac{x}{y^2} + \frac{1}{xy} \tag{8}$$

$$= [11] + [00]\,. \tag{9}$$

Lastly, we can use the rules above from which it immediately follows that [10][01] = [11] + [00].

We mentioned that the charge under the centre of the groups is conserved. For the groups we are dealing with, we have

$$SO(2n+1): \quad \mathbb{Z}_2 \tag{10}$$

$$SO(0 \mod 4): \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{11}$$

$$SO(2 \mod 4): \quad \mathbb{Z}_4\,, \tag{12}$$

technically these are not the correct centres for $SO(n)$ groups but rather of the $Spin(n)$ groups.

Symmetric products

Other important operations are **symmetric** and **anti-symmetric products**. We will begin here with the symmetric product. Suppose we have n variables x_i with $i = 1, \dots, n$. We can combine them to form a symmetric bilinear form

$$a_{ij} = x_i x_j\,, \tag{13}$$

this will then have

$$\begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \frac{n(n+1)}{2} \tag{14}$$

independent elements, corresponding to the indices $i \leq j$. We can generalise this to a fully symmetric rank-k tensor

$$a_{i_1 \dots i_k} = x_{i_1} \dots x_{i_k} \, . \tag{15}$$

The indices corresponding to independent elements are then $i_1 \leq i_2 \leq \dots \leq i_k$, and as such the number of elements is

$$\begin{pmatrix} n+k-1 \\ k \end{pmatrix} . \tag{16}$$

We can construct a generating series for symmetric products. Consider the product

$$\prod_{i=1}^{n} \frac{1}{1-x_i} = (1 + x_1 + x_1^2 + \dots) \dots (1 + x_n + x_n^2 + \dots) \tag{17}$$

$$= 1 + \sum_i x_i + \sum_{i \leq j} x_i x_j + \sum_{i \leq j \leq k} x_i x_j x_k + \dots \, , \tag{18}$$

where we have grouped together all monomials of the same order. If we now set all $x_i = x$, we recover the number of independent elements of a fully symmetric rank-k tensor

$$\frac{1}{(1-x)^n} = 1 + nx + \begin{pmatrix} n+1 \\ 2 \end{pmatrix} x^2 + \dots + \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k + \dots \, . \tag{19}$$

As a concrete example, let us look at symmetric products of the [1] representation of $SU(2)$. The character of this representation is $x + 1/x$, and we can compute symmetric products as

$$\begin{aligned} \frac{1}{1-xt}\frac{1}{1-t/x} &= 1 + (x + \frac{1}{x})t + (x^2 + 1 + \frac{1}{x^2})t^2 + \cdots \\ &= \sum_{n=0}^{\infty} \mathrm{Sym}^n[1] t^n \\ &= \sum_{n=0}^{\infty} [n] t^n \, , \end{aligned} \tag{20}$$

where the second line follows by construction and the third line comes from recognising the characters of $SU(2)$. Similarly, we could compute symmetric products of $[2] = x^2 + 1 + x^{-2}$

$$\begin{aligned} \frac{1}{1-x^2t}\frac{1}{1-t}\frac{1}{1-t/x^2} &= (1 + x^2t + x^4t^2 + \cdots)(1 + t + t^2 + \ldots)(1 + x^{-2}t + x^{-4}t^2 + \cdots) \\ &= 1 + (x^2 + 1 + x^{-2})t + (x^4 + x^2 + 1 + 1 + x^{-2} + x^{-4})t^2 + \cdots \\ &= 1 + [2]t + ([4] + [0])t^2 + \cdots . \end{aligned} \tag{21}$$

Aside: Plethystic exponential for symmetric products

We can be slightly more explicit and formal about how we calculate the symmetric products by introducing the **plethystic exponential**. Consider the following

$$\begin{aligned} \prod_{i=1}^{n} \frac{1}{1-x_i} &= \exp\left(\log\left(\prod_{i=1}^{n} \frac{1}{1-x_i}\right)\right) \\ &= \exp\left(-\sum_{i=1}^{n} \log(1-x_i)\right) \\ &= \exp\left(\sum_{i=1}^{n}\sum_{k=1}^{\infty} \frac{x_i^k}{k}\right), \end{aligned} \tag{22}$$

where, to get to the last line, we used the relation

$$\log(1-x_i) = -\sum_{k=1}^{\infty} \frac{x_i^k}{k}, \tag{23}$$

which can be derived from integrating $1/(1-x_i)$. By now defining

$$f(x_i) = \sum_{i=1}^{n} x_i^k, \tag{24}$$

we get our final expression for the plethystic exponential, where we have to restrict $f(0) = 0$,

$$\mathrm{PE}[f(x_i)] = \exp\left(\sum_{k=1}^{\infty} \frac{f(x_i)}{k}\right). \tag{25}$$

If we also include a counting fugacity into the argument of the plethystic exponential, we get the formal expression of what we have already seen above with the characters

$$\mathrm{PE}[f(x_i)t] = \sum_{k=0}^{\infty} \mathrm{Sym}^k[f]\, t^k. \tag{26}$$

Anti-symmetric product

We can follow a similar procedure to obtain the results for antisymmetric products; here, we will simply state the final answers. The number of independent elements for a fully antisymmetric rank-k tensor is

$$\binom{n}{k}\,, \tag{27}$$

corresponding to the indices $i_1 < i_2 < \cdots < i_k$. The generating function thus becomes

$$\begin{aligned}\prod_{i=1}^{n}(1-x_i) &= 1+\sum_i x_i + \sum_{i<j} x_i x_j + \cdots \\ &= \sum_{k=0}^{n}\binom{n}{k} x^k\,,\end{aligned} \tag{28}$$

where we have substituted $x_i = x$ in the last line. The plethystic exponential now takes the form

$$\mathrm{PE}_F[f(x_i)] = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}f(x_i)}{k}\right), \tag{29}$$

and with a counting fugacity

$$\mathrm{PE}_F[f(x_i)t] = \sum_{k=0}^{n}\Lambda^k[f]\,t^k\,. \tag{30}$$

2.2 $SO(n)$ representations

A useful way to think about the representations is in terms of Dynkin diagram nodes. The fundamental weights are defined as the dual space to the fundamental roots as

$$\lambda_i(\alpha_j^\vee) = \delta_{ij} \tag{31}$$

where $\alpha_i^\vee$ is the coroot. Since each node in a Dynkin diagram corresponds to a root, it can also be associated with a weight.

For $SO(2n)$, the Lie algebra is D_n, and we have the diagram in figure 2. The names can be understood by looking at the dimensions of representations. We may summarise all this as well as the corresponding field in a table, see table 1. It is a valuable exercise to write down a similar diagram and table for B_n. The dimensions of these representations can often be deduced in some way, but if all methods fail, one can always go back to the Weyl dimensions formula, for which we have computed explicit expression in the appendix A.

There are two relations which are very useful for calculating symmetric and antisymmetric products of spinors of $SO(n)$, they are

$$\mathrm{Sym}^2[0\cdots 01]_{B_n} = \sum_{k=n \bmod 4}\Lambda^k[10\cdots 0]\,, \tag{32}$$

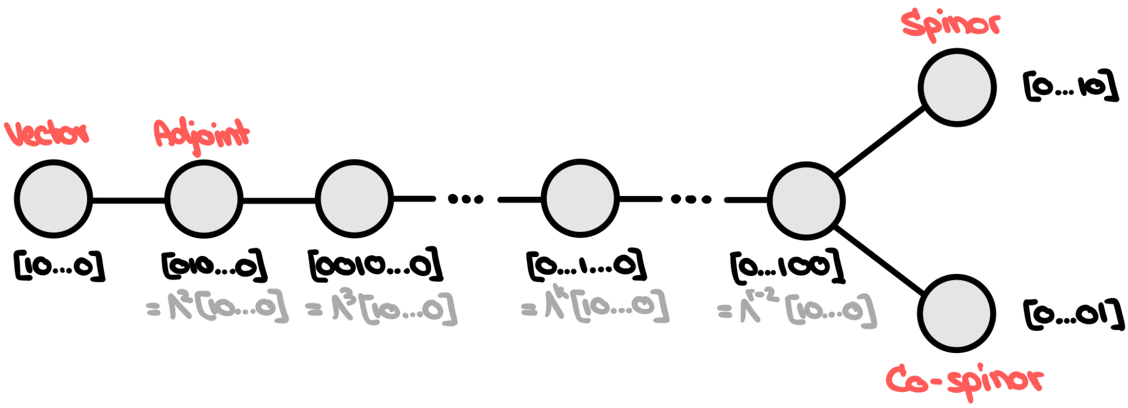


Figure 2: Dynkin diagram for D_n with associated base representations labelled.

D_n with $n \geq 3$	Dynkin Label	Dimension	Field
Scalar	$[0 \cdots 0]$	1	ϕ
Vector	$[10 \cdots 0]$	$2n$	A_μ
Adjoint	$[010 \cdots 0]$	$n(2n-1)$	$B_{\mu_1 \mu_2}$
$\Lambda^k[10 \cdots 0]$	$\Lambda^k[10 \cdots 0]$	$\binom{2n}{k}$	$C_{\mu_1 \cdots \mu_k}$
Spinor	$[0 \cdots 10]$	2^{n-1}	ψ_α
Co-spinor	$[0 \cdots 01]$	2^{n-1}	$\bar{\psi}_{\dot{\alpha}}$

Table 1: Basic representations of D_n. Here μ_i denotes spacetime indices and α denotes spinor indices. Fields with multiple μ indices are antisymmetric.

and

$$\Lambda^2[0 \cdots 01]_{B_n} = \sum_{k=n+2 \bmod 4} \Lambda^k[10 \cdots 0] \,. \tag{33}$$

Both B_n and D_n? As well as a general relation for antisymmetric products of a sum

$$\Lambda^n(R_1 + R_2) = \sum_{k=0}^{n} (\Lambda^n R_1) \left(\Lambda^{n-k} R_2\right) \,. \tag{34}$$

2.3 Example: Supersymmetry algebra

The Lorentz symmetry in 11d is $SO(1,10)$, and thus, we already know that the spinor has dimension $2^5 = 32$ we can then write Q_α with $\alpha = 1, ..., 32$. The most general form of the supersymmetry algebra $\{Q_\alpha, Q_\beta\}$ can easily be calculated by taking the symmetric product of the spinor representation because the anticommutator is invariant under the exchange of Q_α

and Q_β, we have

$$\begin{aligned}\mathrm{Sym}^2[00001] &= \sum_{n=1 \bmod 4} \Lambda^n[10000] \\ &= (\Lambda^1+\Lambda^5+\Lambda^2)[10000] \\ &= [10000]+[00002]+[01000]\end{aligned} \tag{35}$$

where we have used the fact that $\Lambda^9 = \Lambda^2$ in 11d.

This expression can then be translated back into the language of fields

$$\{Q_\alpha, Q_\beta\} = \Gamma^\mu_{\alpha\beta} P_\mu + \Gamma^{\mu_1 \dots \mu_5}_{\alpha\beta} Z_{\mu_1 \dots \mu_5} + \Gamma^{\mu_1\mu_2}_{\alpha\beta} Z_{\mu_1\mu_2}\,, \tag{36}$$

where P_μ is the momentum, $Z_{\mu_1 \dots \mu_n}$ are the central charges and we define

$$\Gamma^{\mu_1 \dots \mu_n}_{\alpha\beta} = \Gamma^{[\mu_1}_{\alpha\tau_1} \dots \Gamma^{\mu_n]}_{\tau_n\beta} \epsilon^{\tau_1\tau_2} \dots \epsilon^{\tau_{n-1}\tau_n}\,. \tag{37}$$

3 Theories with 32 Supercharges

3.1 11d supergravity

The theory of 11d supergravity is the low energy limit of M-theory, and as we will generally be interested in low energy dynamics, we can focus on massless states in the theory. In 11d, we can always boost to a frame where we have $P_\mu = (E, E, 0, ..., 0)$ with $\mu = 0, ..., 10$. From this, we can tell that the little group is $SO(9)$. We have already seen that the spinor dimension in 11d is 32, which dictates the minimum number of supercharges, and we will have $2^{32/4} = 2^8 = 256$ degrees of freedom.

The so-called gravity multiplet G_{11}, which is the only multiplet possible in 11d, is given by

$$G_{11} = \underbrace{\underset{44}{[2000]_9}}_{\text{Graviton}} + \underbrace{\underset{128}{[1001]_9}}_{\text{Gravitino}} + \underbrace{\underset{84}{[0010]_9}}_{\text{3-Form}}\,, \tag{38}$$

where we have annotated the dimension and what type of particle the representations correspond to. We can clearly see that the bosonic degrees of freedom match the fermionic degrees of freedom, as is required for supersymmetry to be preserved.

3.2 Type IIA and type IIB theories

The way to get the type IIA multiplet in 10d is via dimensional reduction from the 11d supergravity multiplet. The idea begin **dimensional reduction** is that if we have a vector in d dimensions, this will turn into a vector and a scalar in $(d-1)$ dimensions. In terms of representations, we have the following when going from 11d to 10d

$$\begin{aligned} \underset{1}{[0000]_9} &\to \underset{1}{[0000]_8} \\ \underset{9}{[1000]_9} &\to \underset{8}{[1000]_8} + \underset{1}{[0000]_8} \\ \underset{16}{[0001]_9} &\to \underset{8}{[0001]_8} + \underset{8}{[0010]_8}\,, \end{aligned} \tag{39}$$

where the $SO(9)$ spinor representation decomposes into both spinors of the $SO(8)$ representation. All other decompositions can be derived from this.

One finds that the 10d **type IIA multiplet** is

$$G_{\text{IIA}} = \underbrace{\underset{35}{[2000]_8}}_{\text{Graviton}} + \underbrace{\underset{28}{[0100]_8}}_{\text{2-Form}} + \underbrace{\underset{1}{[0000]_8}}_{\text{Dilaton}} + \underbrace{\underset{56}{[1001]_8}}_{\text{Gravitino}} + \underbrace{\underset{8}{[0010]_8}}_{\text{Spinor}} + \underbrace{\underset{56}{[1010]_8}}_{\text{Co-Gravitino}} + \underbrace{\underset{8}{[0001]_8}}_{\text{Co-Spinor}} + \underbrace{\underset{56}{[0011]_8}}_{\text{3-Form}} + \underbrace{\underset{8}{[1000]_8}}_{\text{1-Form}}\,. \tag{40}$$

We can note a few points about this. Firstly, the theory is non-chiral, meaning that it includes spinors of both types. And very importantly, it can be factorised into

$$G_{\text{IIA}} = ([1000]_8 + [0001]_8)([1000]_8 + [0010]_8)\,, \tag{41}$$

which can be interpreted as left and right moving modes on a closed string.

> **Exercise 1.** Confirm that (41) does indeed give you (40). (*Hint:* Each representation is charged under the centre of the group $Z(SO(8)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and this charge is conserved under the tensor product.)

The multiplet for **type IIB** is the chiral equivalent of (41)

$$\begin{aligned} G_{\text{IIB}} &= ([1000]_8 + [0001]_8)([1000]_8 + [0001]_8) \\ &= \underbrace{\underset{35}{[2000]_8}}_{\text{Graviton}} + \underbrace{\underset{28}{[0100]_8}}_{\text{2-Form}} + \underbrace{\underset{1}{[0000]_8}}_{\text{Dilaton}} + 2\underbrace{\underset{56}{[1001]_8}}_{\text{Gravitino}} + 2\underbrace{\underset{8}{[0010]_8}}_{\text{Co-Spinor}} + \underbrace{\underset{35}{[0002]_8}}_{\text{Self-Dual 4-Form}} + \underbrace{\underset{1}{[0000]_8}}_{\text{Axion}} + \underbrace{\underset{28}{[0100]_8}}_{\text{2-Form}} \end{aligned} \tag{42}$$

where the other choice of spinor gives a physically equivalent theory. A final note on this is that in the case of type IIB we have 2 scalars which we are free to rotate into each other, this means that there is an $SO(2)$ R-symmetry.

3.3 Lower dimensional theories

We can continue the process of reducing the dimensions. And since the spinors of $SO(8)$ will reduce to the same (and only) spinor of $SO(7)$ it does not matter if we reduce type IIA or type IIB, we will get the same multiplet in 9d. So we get that the lower dimensional multiplets are

$$\begin{aligned} G_9 &= \underset{27}{[200]_7} + \underset{3\times 7}{3[100]_7} + \underset{3\times 1}{3[000]_7} + \underset{2\times 21}{2[010]_7} + \underset{2\times 48}{2[101]_7} + \underset{4\times 8}{4[001]_7} + \underset{35}{[002]_7} \\ G_8 &= \underset{20}{[020]_6} + \underset{6\times 6}{6[010]_6} + \underset{7\times 1}{7[000]_6} + \underset{3\times 15}{3[101]_6} + \underset{2\times 20}{2[110]_6} + \underset{6\times 4}{6[001]_6} + \underset{2\times 20}{2[011]_6} + \underset{6\times 4}{6[100]_6} + \underset{10}{[200]_6} + \underset{10}{[002]_6} \\ G_7 &= \underset{14}{[20]_5} + \underset{10\times 5}{10[10]_5} + \underset{14\times 1}{14[00]_5} + \underset{5\times 10}{5[02]_5} + \underset{4\times 16}{4[11]_5} + \underset{16\times 4}{16[01]_5} \\ G_6 &= \underset{9}{[22]_4} + \underset{16\times 4}{16[11]_4} + \underset{25\times 1}{25[00]_4} + \underset{5\times 3}{5[20]_4} + \underset{5\times 3}{5[02]_4} + \underset{4\times 6}{4[21]_4} + \underset{4\times 6}{4[12]_4} + \underset{20\times 2}{20[01]_4} + +\underset{20\times 2}{20[10]_4} \\ G_5 &= \underset{5}{[4]_3} + \underset{17\times 3}{17[2]_3} + \underset{42\times 1}{42[0]_3} + \underset{10\times 3}{10[2]_3} + \underset{8\times 4}{8[3]_3} + \underset{48\times 2}{48[1]_3}\,, \end{aligned} \tag{43}$$

where for G_8 and G_6 we use A_n Dynkin labels since $SO(6) \simeq A_3$ and $SO(4) \simeq A_1 \times A_1$. When continuing to the 4d case, the little group becomes abelian and we will use fugacities to keep track of charges, so

$$\begin{aligned}[][2]_3 &= q^2 + q^{-2} + 1 \\ [1]_3 &= q + q^{-1}\,. \end{aligned} \tag{44}$$

One might notice that this is equivalent to writing out the character of the $SO(3)$ representations. We get that the gravity multiplet becomes

$$G_4 = q^4 + 8q^3 + 28q^2 + 56q^1 + 70 + 56q^{-1} + 28q^{-2} + 8q^{-3} + q^{-4}\,. \tag{45}$$

And in going down to G_3, all vector quantities become scalars so we only distinguish between scalar (even power of q) and spinor (odd power of q)

$$G_3 = 128 + 128q\,. \tag{46}$$

Another important point is that because the spinors of $SO(8)$ reduce to the same (and only) type of spinor in $SO(7)$, it does not matter if we reduce type IIA or type IIB, we will get the same multiplet in 9d.

Exercise 2. Explicitly derive all G_d for $3 \leq d \leq 10$ from the 11d multiplet.

3.4 Moduli spaces

Scalars in quantum field theories admit a vacuum expectation value (vev), the possible values of the vevs give us the so-called **moduli space** of vacua. For theories with 32 supercharges, the moduli spaces are intricately linked to the exceptional algebra E_n. We begin by listing the number of scalars in the gravity multiplet in each dimension, which immediately follows from our previous discussion on dimension reduction, see table 2. The scalars give us degrees of freedom in the moduli space and, hence, tell us the dimension of the moduli space.

Theory:	11d	Type IIA	Type IIB	9d	8d	7d	6d	5d	4d	3d
Scalars:	0	1	2	3	7	14	25	42	70	128

Table 2: Number of scalars in theories with 32 supercharges.

The next step is to pair every theory in d dimensions to an exceptional algebra E_n such that $n + d = 11$, see table 3 and figure 3.

The moduli space is then given by

$$\mathcal{M} = \frac{E_{n(n)}}{H_n}\,, \tag{47}$$

Theory:	11d	Type IIA	Type IIB	9d
E_n Algebra:	$E_0 = \emptyset$	$\tilde{E}_1 = U(1)$	$E_1 = SU(2)$	$E_2 = U(1) \times SU(2)$

Table 3: Abelian exceptional algebras corresponding to theories in dimensions 11 to 9.

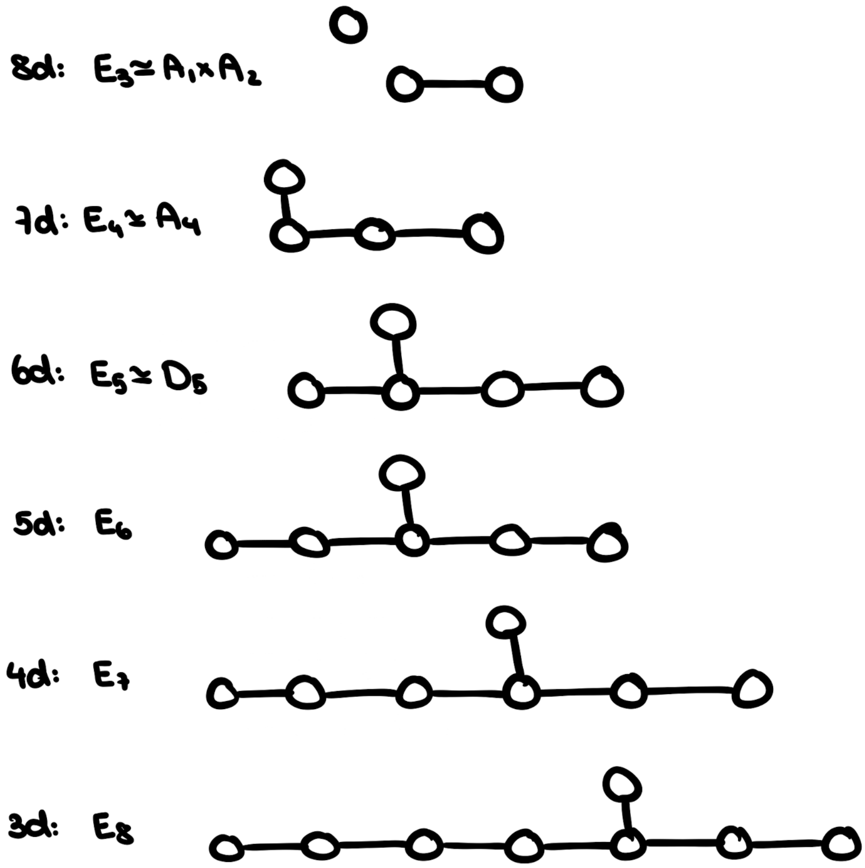


Figure 3: Exceptional algebras corresponding to theories in dimensions 8 to 3.

where $E_{n(n)}$ is the maximally non-compact group with algebra E_n, the first n in the subscript denotes the rank and the n in brackets denotes the number of non-compact generators minus the number of compact generators, and H_n is the maximally compact subgroup of $E_{n(n)}$.

The split form (or maximally non-compact group) $E_{n(n)}$ consists of both compact and non-compact generators. For the moduli space, we factor out the compact generators and are left with the non-compact generators, which correspond to the scalar fields in our theory. We have that

$$\dim\, E_{n(n)} = \underbrace{\dim\, H_n}_{\text{compact}} + \underbrace{\dim\, E_{n(n)}/H_n}_{\text{non-compact}} \,. \tag{48}$$

Another interesting feature is that H_n corresponds to the R-symmetry.

For example, the 8d theory corresponds to the algebra $E_3 \simeq A_1 \times A_2$. The maximally non-compact group with this algebra is $SL(2, \mathbb{R}) \times SL(3, \mathbb{R})$, and its maximally compact subgroup is $SO(2)\times SO(3)$. Hence, the moduli space is

$$\mathcal{M}_8 = \frac{SL(2,\mathbb{R})\times SL(3,\mathbb{R})}{SO(2)\times SO(3)}\,. \tag{49}$$

And the dimension of this space is

$$\dim \mathcal{M}_8 = 3+8-1-3 = 7\,, \tag{50}$$

which matches the number of scalars that we found before. The number of non-compact generators is 7 and the number of compact generators is 4, which correctly gives us

$$\dim E_{3(3)} = (\text{non-compact}) + (\text{compact}) = 11 \tag{51}$$

$$(3) = (\text{non-compact}) - (\text{compact}) = 3\,. \tag{52}$$

We extend this analysis to the remaining theories in table 4.

Theory	Moduli Space $\mathcal{M}_n$	Dimension = Non-compact Gen.	Compact Gen.
10d Type IIA	$\mathbb{R}^+$	1	0
10d Type IIB	$SL(2,\mathbb{R})/SO(2)$	2	1
9d	$\mathbb{R}^+\times SL(2,\mathbb{R})/SO(2)$	4	2
8d	$\frac{SL(2,\mathbb{R})\times SL(3,\mathbb{R})}{SO(2)\times SO(3)}$	7	4
7d	$SL(5,\mathbb{R})/SO(5)$	14	10
6d	$SO(5,5)/SO(5)\times SO(5)$	25	20
5d	$E_{6(6)}/Sp(4)$	42	36
4d	$E_{7(7)}/SU(8)$	70	63
3d	$E_{8(8)}/SO(16)$	128	120

Table 4: Moduli spaces of theories with 32 supercharges.

4 Theories with 16 Supercharges

4.1 Type I and Heterotic theories

In the following, we will construct multiplets for the heterotic theories and only mention in passing that there exists another theory, type I, with 16 supercharges in 10 dimensions. We will properly deal with type I once we have introduced branes and orientifolds.

For theories with 32 supercharges, the minimal multiplet has $2^{32/4} = 256$ degrees of freedom, corresponding to the supergravity multiplet. Similarly, the minimal supermultiplet in a theory with 16 supercharges has $2^{16/4} = 16$ degrees of freedom.

Since we also want to preserve supersymmetry, we need an equal number of bosonic and fermionic degrees of freedom, the vector multiplet (vplet) in 10d satisfies this and is

$$\underset{16}{V_{10}} = \underset{8}{[1000]_8} + \underset{2^{4-1}=8}{[0001]_8} \,. \tag{53}$$

To get a graviton, and thus the gravity multiplet, we simply need to tensor product this with $[1000]_8$

$$\begin{aligned} G_{10} &= V_{10}[1000]_8 \\ &= \underset{64}{\overset{(0,0)}{[1000]_8[1000]_8}} + \underset{64}{\overset{(0,1)}{[0001]_8[1000]_8}} \\ &= \underbrace{\underset{\text{Graviton}}{\underset{35}{[2000]_8}} + \underset{\text{2-Form}}{\underset{28}{[0100]_8}} + \underset{\text{Dilaton}}{\underset{1}{[0000]_8}}}_{\text{NS-NS sector}} + \underbrace{\underset{\text{Gravitino}}{\underset{56}{[1001]_8}} + \underset{\text{Gravifermion}}{\underset{8}{[0010]_8}}}_{\text{R-NS sector}} \,, \end{aligned} \tag{54}$$

where we have as usual denoted the dimensions, as well as the charge under the centre of $SO(8)$ i.e. the charge under $\mathbb{Z}_2 \times \mathbb{Z}_2$. We may note at this point that because there is no R-R sector, there cannot be any D branes (this statement will make more sense once we have actually dealt with branes). The interpretation of the above equation is that we have a closed string whose left (right) moving modes are supersymmetric and whose right (left) moving modes are purely bosonic.

The gauge field in the vector multiplet admits a Yang-Mills extension, and we may write this as

$$V \times (\text{adjoint of gauge group}) = VA \,. \tag{55}$$

The massless sector of the heterotic theories is

$$VA + G = \underbrace{V}_{\text{left moving}} \times \underbrace{(A + [1000]_8)}_{\text{right moving}} \,. \tag{56}$$

Additionally, to ensure that the theory is anomaly free we need the gauge group to be of dimension 496. The only groups of this dimension are $E_8 \times E_8$ and $SO(32)$. The resulting theories are thus

- **Heterotic SO(32)**
- **Heterotic $\mathbf{E_8} \times \mathbf{E_8}$.**

4.2 Lower dimensional theories

Similar to the discussion on dimensional reduction for theories with 32 supercharges, we can dimensionally reduce the multiplets (53) and (54). One finds that in general for $5 \leq d \leq 10$

$$\begin{aligned} V_d &\mapsto V_{d-1} \\ G_d &\mapsto G_{d-1} + V_{d-1}\,, \end{aligned} \tag{57}$$

as well that the gravity multiplet G_d includes only a single scalar for $4 \leq d \leq 10$.

Explicitly calculating the lower dimensional vector multiplets, we find

$$\begin{aligned} V_9 &= [100]_7 + [000]_7 + [001]_7 \\ V_8 &= [010]_6 + 2[000]_6 + [100]_6 + [001]_6\,, \end{aligned} \tag{58}$$

since there are 2 scalars in V_8 we can rotate them into each other, giving us an external $SO(2)$ symmetry, the R-symmetry. To keep track of the charge under $SO(2)$, we can assign fugacities and write

$$V_8 = [010]_6 + [100]_6 q^1 + [001]_6 q^{-1} + [000]_6 (q^2 + q^{-2}). \tag{59}$$

Continuing this spiel, for V_7

$$\begin{aligned} V_7 &= [10]_5 + 2[01]_5 + 3[00]_5 \\ &= [10]_5[00]_R + [01]_5[1]_R + [00]_5[2]_R\,, \end{aligned} \tag{60}$$

where we have recognised the R-symmetry to be $SO(3) \simeq A_1$ and labelled the R-symmetry representation with a subscript R. The remaining vector multiplets are

$$\begin{aligned} V_6 &= [11]_4[00]_R + [10]_4[10]_R + [01]_4[01]_R + [00]_4[11]_R \\ V_5 &= [2]_3[00]_R + [1]_3[01]_R + [0]_3[10]_R \\ V_4 &= (q^2 + q^{-2})[000]_R + q[100]_R + q^{-1}[001]_R + q^0[010]_R \\ V_3 &= q[0001]_R + [1000]_R\,. \end{aligned} \tag{61}$$

One might notice that for V_3 the number of scalars increases by 2 to 8, so the R-symmetry is $SO(8)$.

Another important aspect is that because the gauge group is of rank 16, we will get an additional 16 massless vector multiplets upon dimensional reduction. So, starting in 10 dimensions, we have

$$G_{10} \to G_9 + 17V_9 \to G_8 + 18V_8 \to G_7 + 19V_7 \to \cdots . \tag{62}$$

For $5 \leq d \leq 9$ the moduli space turns out to be

$$\mathcal{M}_d = \frac{SO(n, n-16)}{SO(n) \times SO(n-16)} \times \mathbb{R}^+ , \tag{63}$$

with special cases

$$\begin{aligned} \mathcal{M}_{10} &= \mathbb{R}^+ \\ \mathcal{M}_4 &= \frac{SO(22,6)}{SO(22) \times SO(6)} \times \frac{SL(2,\mathbb{R})}{SO(2)} \\ \mathcal{M}_3 &= \frac{SO(24,8)}{SO(24) \times SO(8)} , \end{aligned} \tag{64}$$

with the second factor for $\mathcal{M}_4$ reflecting the fact that G_4 now has 2 scalars rather than just 1, and for $\mathcal{M}_3$ G_3 completely breaks down??.

4.3 Tensor multiplet

This subsection may be skipped on a first reading and can be returned to after section 5 on branes.

We have already seen the vector multiplet as the supersymmetric multiplet in theories with 16 supercharges in dimensions 3 to 10. There turns out to be another type of multiplet in 6 dimensions, the so-called **tensor multiplet**, which we will construct in this section.

Consider first a 5d theory with 16 supercharges, which we can build by putting a D4 brane in Type IIA. The massless multiplets will thus fall into representations of

$$\underbrace{SO(3)}_{\text{Little Group on D4 Brane}} \times \underbrace{SO(5)}_{\text{R-Symmetry}} \subset \underbrace{SO(8)}_{\text{Little Group Spacetime}} . \tag{65}$$

This is the vector multiplet that we have already met in (61)

$$V_5 = [2]_3[00]_R + [1]_3[01]_R + [0]_3[10]_R . \tag{66}$$

The important conceptual point to understand is that the little group for 5d, $SO(3)$, has representations in A_1 and the little group for 6d, $SO(4)$, has representations in $A_1 \times A_1$. As such, we can simply stick a 0 into the Dynkin labels for the little group of V_5 to make a new multiplet for a 6 dimensional theory. We can do this in two ways

$$\begin{aligned} T_{(2,0)} &= [20]_4[00]_R + [10]_4[01]_R + [00]_4[10]_R \\ T_{(0,2)} &= [02]_4[00]_R + [01]_4[01]_R + [00]_4[10]_R\,, \end{aligned} \tag{67}$$

which are the tensor multiplets. We then also have two types of gravity multiplet in 6d, the one we get by dimensional reduction is

$$\begin{aligned} G_{(1,1)} &= V_6[11]_4 \\ &= \Big([11]_4[00]_R + [10]_4[10]_R + [01]_4[01]_R + [00]_4[11]_R\Big)[11]_4 \\ &= \Big([22] + [20] + [02] + [00]\Big)_4[00]_R + \Big([12] + [10]\Big)_4[01]_R \\ &\quad + \Big([21] + [01]\Big)_4[10]_R + [11]_4[11]_R\,, \end{aligned} \tag{68}$$

and the one from the tensor multiplet

$$\begin{aligned} G_{(2,0)} &= T_{(2,0)}[02]_4 \\ &= [22]_4[00]_R + [12]_4[01]_R + [02]_4[10]_R\,. \end{aligned} \tag{69}$$

5 Generalised Electromagnetism (From Forms to Branes)

5.1 Basic principle

Electromagnetism in 4d is described by the familiar action

$$S = \int F^{(2)} \wedge *F^{(2)} + A^{(1)} \wedge *J^{(1)} \,, \tag{70}$$

where $F^{(2)}$ is the field strength associated with the 1-form gauge field $A^{(1)}$, and in the usual scenario without magnetic sources the field strength is given by $F = dA$, and $J^{(1)}$ is the electric current. We then get the usual Maxwell equations by noting that $F^{(2)}$ satisfies the Bianchi identity, as well as calculating the equations of motion

$$dF^{(2)} = 0 \qquad \text{and} \qquad d * F^{(2)} = *J^{(1)} \,. \tag{71}$$

If we want to include magnetic charges, the Maxwell equations (71) become

$$dF^{(2)} = *J^{(1)}_m \qquad \text{and} \qquad d * F^{(2)} = *J^{(1)}_e \,, \tag{72}$$

where $J^{(1)}_m$ and $J^{(1)}_e$ are the magnetic and electric currents, respectively. For a point-like source, we simply replace the currents by a delta function

$$dF^{(2)} = Q_m \delta^{(3)} \qquad \text{and} \qquad d * F^{(2)} = Q_e \delta^{(3)} \,. \tag{73}$$

The next step in the analysis is to generalise this to arbitrary p-forms in D dimensions, this fortunately is very straightforward to do. For a form $C^{(p)}$ there is an associated field strength $G^{(p+1)}$ and the Maxwell equations are

$$dG^{(p+1)} = Q_m \delta^{(p+2)} \qquad \text{and} \qquad d *_D G^{(p+1)} = Q_e \delta^{(D-p)} \,. \tag{74}$$

5.2 Branes in 11d supergravity

The massless multiplet in 11d supergravity is

$$[2000] + [0010] + [1001] \,, \tag{75}$$

from this we can tell that there is 3-form, which we may call $C^{(3)}$ as well as its corresponding field strength $G^{(4)}$. We can now replace $G^{(p+1)}$ in (74) to get

$$dG^{(4)} = Q_m \delta^{(5)} \qquad \text{and} \qquad d *_{11} G^{(4)} = Q_e \delta^{(8)} \,. \tag{76}$$

Here the magnetically charged object is locallised in 5 space dimensions and thus spans over 5 other space dimensions, we call this object the **M5 brane**. On the other hand, the electrically charged object is localised in 8 space dimensions and hence spans over the 2 remaining directions, this is called the **M2 brane**.

5.3 Branes in Type IIA

For theories in lower dimensions, we can proceed in the same fashion. The multiplet in Type IIA is

$$\underbrace{[2000]+[0100]+[0000]}_{\text{NS-NS Sector}}+\underbrace{[1001]+[0010]+[1010]+[0001]}_{\text{NS-R Sector}}+\underbrace{[0011]+[1000]}_{\text{R-R Sector}}\,, \tag{77}$$

where we identify the form $B^{(2)}$ in the NS-NS sector (we are neglecting the dilaton 0-form since it does not source a brane) and the forms $C^{(3)}$, $C^{(1)}$ in the R-R sector.

For the forms in the R-R sector, we may begin with $C^{(1)}$

$$dG^{(2)} = Q_m \delta^{(3)} \qquad \text{and} \qquad d *_{10} G^{(2)} = Q_e \delta^{(9)}\,, \tag{78}$$

by the same logic as for the 11d supergravity case, we have a so-called **D6 brane** and **D0 brane** which function as a magnetic source and an electric source, respectively. Continuing with the $C^{(3)}$, we have

$$dG^{(4)} = Q_m \delta^{(5)} \qquad \text{and} \qquad d *_{10} G^{(4)} = Q_e \delta^{(7)}\,, \tag{79}$$

giving us a **D4 brane** and a **D2 brane**. There is also a non-dynamical **D8 brane**, but contrary to the other branes does not come from a form field in the multiplet. We can already note that in Type IIA we have even Dp branes.

For the NS-NS sector, we only have the $B^{(2)}$ form with field strength $H^{(3)}$

$$dH^{(3)} = Q_m \delta^{(4)} \qquad \text{and} \qquad d *_{10} H^{(3)} = Q_e \delta^{(8)}\,, \tag{80}$$

these objects are called the **NS5 brane** and the **F1 brane**, where the latter is the fundamental string.

5.4 Branes in Type IIB

The multiplet in Type IIB is

$$\underbrace{[2000]+[0100]+[0000]}_{\text{NS-NS Sector}}+\underbrace{2[1001]+2[0010]}_{\text{NS-R Sector}}+\underbrace{[0002]+[0100]+[0000]}_{\text{R-R Sector}}\,, \tag{81}$$

where the NS-NS sector is identically, so the same results as for Type IIA follow, and we have the form fields $C^{(0)}$, $C^{(2)}$ and $C^{(4)+}$. The 4-form corresponds to the self-dual part of $\Lambda^4[1000]$.

Beginning with the $C^{(0)}$ form and its field strength $G^{(1)}$

$$dG^{(1)}=Q_m\delta^{(2)} \qquad \text{and} \qquad d*_{10}G^{(1)}=Q_e\delta^{(10)}\,, \tag{82}$$

where the magnetic source is the **D7 brane** and the electrical source is an object which is both localised in space and in time, this is an **instanton** and is usually referred to as the **D(-1) brane**. For the $C^{(2)}$ we get

$$dG^{(3)}=Q_m\delta^{(4)} \qquad \text{and} \qquad d*_{10}G^{(3)}=Q_e\delta^{(8)}\,, \tag{83}$$

these are the **D5 brane** and the **D1 brane**. The field strength $G^{(5)}$ associated with the self-dual form $C^{(4)}$ is special because in 10d it is itself self-dual $G^{(5)}=*_{10}G^{(5)}$, the consequence is that we will have an object which is both magnetically and electrically charged

$$dG^{(5)}=d*_{10}G^{(5)}=Q_{e/m}\delta^{(6)}\,. \tag{84}$$

We thus say that the **D3 brane** is a **dyonic** object. Here we are again missing a brane, namely the **D9 brane**, since it is space filling, it is non-dynamical. The importance of the D9 brane will become clearer in section 7.

5.5 More properties of branes

We have so far met the M2 and M5 brane in 11d supergravity, as well as the Dp brane with $-1\leq p\leq 9$ from the RR sector and the NS5 and F1 branes from the NS-NS sector. In general, a p brane will break Minkowski space into its own worldvolume and the space surrounding it

$$\underbrace{\mathbb{R}^{1,p}}_{\text{World Volume}}\times\underbrace{\mathbb{R}^{9-p}}_{\text{Point-like}}\subset\underbrace{\mathbb{R}^{1,9}}_{\text{Spacetime}}\,. \tag{85}$$

Because we are breaking a continuous symmetry, there will be $9-p$ Goldstone modes living on the p brane. Additionally, branes break half the supersymmetry on their world volume (32 supercharges to 16 supercharges), resulting in 8 goldstinos. At first, this might seem like a problem because we already know that there should be 8 bosonic degrees of freedom and 8 fermionic degrees of freedom. To remedy this, we will have to consider branes ending on branes in the next section.

6 Branes Ending on Branes

6.1 F1 brane ending on Dp brane

For this first case of branes ending on branes, we will give a pedagogical introduction to what happens when we spatially restrict branes to other branes. We recall that in Type IIA/IIB we have an F1 brane which is electrically charged under a 2-form $B^{(2)}$

$$d *_{10} H^{(3)} = Q_e \delta^{(8)} \,, \tag{86}$$

where $H^{(3)}$ is the field strength of $B^{(2)}$.

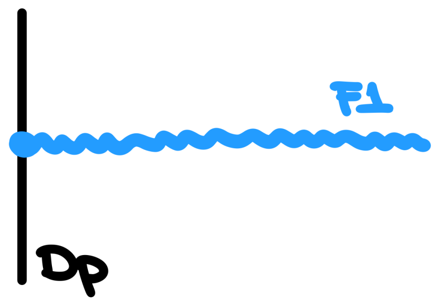


Figure 4: F1 brane ending on a Dp brane.

We now assume that the F1 brane ends on some Dp brane, see figure 4. Naively, we can write this as

$$d *_{10} H^{(3)} = \delta^{(8)} \theta \,, \tag{87}$$

where we set the charge to $Q_e = 1$ and θ is the Heaviside function. But by taking the exterior derivative d of both sides, we quickly see that we are missing something

$$\begin{aligned} dd *_{10} H^{(3)} &= \delta^{(8)} d\theta \\ 0 &= \delta^{(9)} \,, \end{aligned} \tag{88}$$

which is obviously wrong. The way to remedy this problem is to think about what the F1 brane would look like from the perspective of someone living on the Dp brane. Since the F1 brane is 2 dimensional, its boundary is 1 dimensional and thus an observer on the brane would see a 1 dimensional and electrically charged object. Or, in the form of an equation

$$d *_{p+1} F^{(2)} = \delta^{(p)} \,. \tag{89}$$

This can be incorporated into (87) as

$$d *_{10} H^{(3)} = \delta^{(8)} \theta - \delta^{(9-p)} *_{p+1} F^{(2)} \,, \tag{90}$$

and, by again taking the exterior derivative d, we reproduce (89).

The next step is to write down an action which will give us (90), again taking the naive approach we can write

$$\begin{aligned} S &= \int_{\mathbb{R}^{1,9}} H^{(3)} \wedge * H^{(3)} + \int_{\mathbb{R}^{1,9}} B^{(2)} \wedge \delta^{(9-p)} *_{p+1} F^{(2)} - \int_{\mathbb{R}^{1,9}} B^{(2)} \wedge \delta^{(8)} \theta \\ &= \underbrace{\int_{\mathbb{R}^{1,9}} H^{(3)} \wedge * H^{(3)}}_{\text{Spacetime}} + \underbrace{\int_{\mathbb{R}^{1,p}} B^{(2)} \wedge *_{p+1} F^{(2)}}_{\text{World-volume D}p\text{ Brane}} - \underbrace{\int_{\mathbb{R}^{1,1}} B^{(2)} \theta}_{\text{World-sheet F1 Brane}} \,. \end{aligned} \tag{91}$$

The problem we now have is that this is not fully gauge invariant. The gauge transformations with respect to $B^{(2)}$ are

$$\delta_B B^{(2)} = d\Lambda^{(1)}, \qquad \delta_B H^{(3)} = 0, \qquad \delta_B A^{(1)} = \Lambda^{(1)}, \qquad \text{and} \qquad \delta_B F^{(2)} = d\Lambda^{(1)} \,. \tag{92}$$

Clearly, the spacetime term in the action (91) is already gauge invariant, but the other two are not. If we look at the variation of the world-sheet term, we get

$$\delta_B \int_\Sigma B^{(2)} \theta = \int_\Sigma d\Lambda^{(1)} = \int_{\partial\Sigma} \Lambda^{(1)} \,, \tag{93}$$

where we denote the world-sheet of F1 as Σ. If we include an additional term in the action of the form

$$\int_{\partial\Sigma} A^{(1)} \,, \tag{94}$$

one can see that the combination is then gauge invariant. To ensure the remaining term is gauge invariant, we define $\mathcal{F}^{(2)} = F^{(2)} - B^{(2)}$ such that $\delta_B \mathcal{F}^{(2)} = d\Lambda^{(1)} - d\Lambda^{(1)} = 0$. Therefore the gauge invariant action can be written as

$$S = \int_{\mathbb{R}^{1,9}} H^{(3)} \wedge * H^{(3)} + \int_{\mathrm{D}p} \mathcal{F}^{(2)} \wedge *_{p+1} \mathcal{F}^{(2)} - \int_\Sigma B^{(2)} \theta + \int_{\partial\Sigma} A^{(1)} \,, \tag{95}$$

and this also means that the equation of motion is

$$d *_{10} H^{(3)} = \delta^{(8)} \theta - \delta^{(9-p)} *_{p+1} \mathcal{F}^{(2)} \,. \tag{96}$$

6.2 M2 brane ending on M5 brane

We can extend our analysis to branes in 11d supergravity. Recall that we have a magnetic M5 brane and an electric M2 brane. The M2 brane can end on the M5 brane. We can again ask what the M2 brane looks like from the perspective of an observer on the M5 brane. Because the boundary of the M2 brane is a 2 dimensional, electrically charged object, we have

$$d *_6 F^{(3)} = \delta^{(4)} \,, \tag{97}$$

which implies that there exists a 2-form $A^{(2)}$ on the world-volume of the M5 brane. Proceeding in the same way as with the F1 brane ending on a Dp brane, we get the following source equation for an M2 brane ending on an M5 brane

$$d *_{11} G^{(4)} = \delta^{(8)}\theta - \delta^{(4)} *_6 \mathcal{F}^{(3)}\,, \tag{98}$$

where we again have made the field strength $F^{(3)}$ gauge invariant and wrote it as $\mathcal{F}^{(3)}$.

6.3 Dp branes ending on NS5 brane in type IIA

In Type II theories we can have Dp branes ending on NS5 branes for $0 \leq p \leq 6$. We will begin with branes in Type IIA before moving on to Type IIB. As discussed in section 5.3,

- The D6 brane carries a magnetic charge and the D0 brane an electric charge under the 1-form $C^{(1)}$
- The D4 brane carries a magnetic charge and the D2 brane an electric charge under the 3-form $C^{(3)}$.

D0 ending on NS5 The boundary of the D0 brane is a 0 dimensional, electrically charged object. From the NS5 brane it looks like

$$d *_6 F^{(1)} = \delta^{(6)}\,. \tag{99}$$

We therefore have a 0-form $A^{(0)}$ on the world-volume on the NS5 brane, the source equation becomes

$$d *_{10} G^{(2)} = \delta^{(9)}\theta - \delta^{(4)} *_6 \mathcal{F}^{(1)}\,. \tag{100}$$

Since we have a 0-form $A^{(0)}$, we could also have a magnetically charged object under it. This would look like

$$dF^{(1)} = \delta^{(2)}\,, \tag{101}$$

which is some 3 brane, of course, this will turn out to be the boundary of a D4 brane ending on NS5.

D4 ending on NS5 The boundary of the D4 brane is a 4 dimensional, magnetically charged object and is, as we have already seen,

$$dF^{(1)} = \delta^{(2)}\,. \tag{102}$$

The source equation looks slightly different now, reflecting the fact that D4 is magnetically charged,

$$dG^{(4)} = \delta^{(5)}\theta - \delta^{(4)}\mathcal{F}^{(1)}\,. \tag{103}$$

D2 ending on NS5 Again, the boundary of D2 is 2 dimensional and electrically charged and thus

$$d *_6 F^{(3)} = \delta^{(4)} . \tag{104}$$

This gives us a self-dual 2-form $A^{(2)+}$, the corresponding magnetically charged object looks like

$$dF^{(3)} = \delta^{(4)} . \tag{105}$$

D6 ending on NS5 The magnetically charged D6 brane is special because its 6d boundary is the entire NS5 brane, as a result, we get that $n_1 = n_2$ for two D6 branes connected to a NS5 brane, see figure 5. This is essentially a statement of conservation of charge.

Figure 5: D6 brane ending on a NS5 brane.

Lastly, let us summarise the form field content that we have found on the NS5 brane from our analysis: A 0-form $A^{(0)}$ and a self-dual 2-form $A^{(2)+}$. To accommodate these forms and looking back to our discussions of multiplets with 16 supercharges in section 4, we know that on the NS5 brane, we must have a tensor multiplet $T_{(2,0)}$.

6.4 Dp branes ending on NS5 brane in type IIB

Moving on to the branes in type IIB, we have

- D7 carrying a magnetic charge and D(-1) carrying an electric charge under $C^{(0)}$
- D5 carying a magnetic charge and D1 carrying an electric charge under $C^{(2)}$
- D3 carying a magnetic and an electric charge under $C^{(4)+}$.

The D7 and D(-1) cannot end on the NS5 brane, for the others we have:

D1 ending on NS5 The boundary of the D1 brane is an electrically charged 1 dimensional object, thus

$$d *_6 F^{(2)} = \delta^{(5)} . \tag{106}$$

This means that D1 ending on NS5 induces a gauge field $A^{(1)}$. The object which would be magnetically charged under this field is

$$dF^{(2)} = \delta^{(3)} . \tag{107}$$

This is a 2 brane, which we know will be the boundary of D3 brane.

D3 ending on NS5 Since the D3 brane is a dyonic object, its 3 dimensional boundary will also have both electric and magnetic charges

$$d *_6 F_1^{(4)} = \delta^{(3)} \qquad \text{and} \qquad dF_2^{(2)} = \delta^{(3)} , \tag{108}$$

giving us a 3-form $A_1^{(3)}$ and a 1-form $A_2^{(1)}$. In 6d, a 3-form is nothing but a 1-form, since $\Lambda^3[11] = [11]$. The object which is magnetically charged under the 3-form looks like

$$dF^{(4)} = \delta^{(5)} , \tag{109}$$

which is a 0 brane, or the boundary of a D1 brane.

D5 ending on NS5 This case is again a little bit special, since the boundary of the D5 brane splits the NS5 brane into 2 halves, we have a domain wall. This is essentially like a capacitor, with charges Q_1 and Q_2 on the respective sides of the NS5 brane and their difference is proportional to the charge of the D5 brane

$$Q_1 - Q_2 \propto Q_{\text{D5 Brane}} . \tag{110}$$

We have found a 1-form on the NS5 brane, as such, we can no longer have the tensor multiplet on NS5 but will rather have the vector multiplet V_6.

6.5 (p, q) strings and branes in type IIB

We have brushed over an important aspect of type IIB theories. As we already know, we have two 2-forms: $C^{(2)}$ in the RR sector (giving rise to D1 and D5) and $B^{(2)}$ in the NS-NS sector (giving rise to F1 and NS5). Suppose that we have an F1 brane ending on D1, as we have seen for the case of D5 ending on NS5, the boundary of the F1 brane acts as a domain wall. Therefore, if we label the charge under the forms as

$$\text{F1: } (\underbrace{1}_{B^{(2)}\text{ Charge}}, \underbrace{0}_{C^{(2)}\text{ Charge}}) \qquad \text{and} \qquad \text{D1: } (\underbrace{0}_{B^{(2)}\text{ Charge}}, \underbrace{1}_{C^{(2)}\text{ Charge}}) , \tag{111}$$

we find that one side of the D1 brane needs to have a charge of $(1, 1)$, see figure 6.

Figure 6: (p, q) strings. Don't think this is right.. maybe n D1 branes give (0,n)?

We can make the same constructions for the 5 branes, D5 and NS5, where we define the charges as

$$\text{NS5: } (\underbrace{0}_{C^{(2)}\text{ Charge}}, \underbrace{1}_{B^{(2)}\text{ Charge}}) \quad \text{and} \quad \text{D5: } (\underbrace{1}_{C^{(2)}\text{ Charge}}, \underbrace{0}_{B^{(2)}\text{ Charge}}). \tag{112}$$

And we have the general rule: A (p, q) string can only end on a (p, q) brane.

6.6 (p, q) string ending on a (p, q) brane

For a free F1 and D1 brane, we have source equations

$$d *_{10} H^{(3)} = p\delta^{(8)} \tag{113}$$

$$d *_{10} G^{(3)} = q\delta^{(8)}, \tag{114}$$

where p and q are the charges under the respective fields. In the space-time picture, we could also write this as a vector equation

$$d *_{10} \begin{pmatrix} H^{(3)} \\ G^{(3)} \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \delta^{(8)}, \tag{115}$$

which is invariant under $SL(2, \mathbb{Z})$.

If the (p, q) string ends on the (p, q) 5-brane, we will see an electric 0-brane which crucially does not depend on p or q as the worldvolume observer has no knowledge of either one, we write

$$d *_6 F^{(2)} = \delta^{(5)}. \tag{116}$$

The source equations are therefore

$$d *_{10} H^{(3)} = p\delta^{(8)}\theta - p\delta^{(4)} \wedge *_6 F^{(2)} \tag{117}$$

$$d *_{10} G^{(3)} = q\delta^{(8)}\theta - q\delta^{(4)} \wedge *_6 F^{(2)}, \tag{118}$$

where the additional factors of p and q in front of the $\delta^{(4)} \wedge *_6 F^{(2)}$ terms ensure that, if we take the exterior derivative, we recover (116) without any p or q dependence.

The next step is to write an action which reproduces the three equations of motion (116),

(117) and (118). An initial first guess would be

$$S = \int_{\mathbb{R}^{1,9}} H^{(3)} \wedge *_{10} H^{(3)} + G^{(3)} \wedge *_{10} G^{(3)} \tag{119}$$

$$+ \int_{(p,q)\text{ 5-brane}} -F^{(2)} \wedge *_6 F^{(2)} + (pB^{(2)} + qC^{(2)}) \wedge *_6 F^{(2)} \tag{120}$$

$$+ \int_{(p,q)\text{ string}} -pB^{(2)} - qC^{(2)} + \int_{\partial(p,q)\text{ string}} A^{(1)} \,. \tag{121}$$

This action is not gauge invariant. To fix this, we first anticipate that we will need to change the 5-brane worldvolume term to

$$\int_{(p,q)\text{ 5-brane}} (pB^{(2)} + qC^{(2)} - F^{(2)}) \wedge *_6 (pB^{(2)} + qC^{(2)} - F^{(2)}) \,, \tag{122}$$

where we will define $\mathcal{F}^{(2)} = pB^{(2)} + qC^{(2)} - F^{(2)}$ from now on. For the gauge variations

$$\delta_B B^{(2)} = d\Lambda^{(1)} \,, \quad \delta_B C^{(2)} = 0 \,, \quad \text{and} \quad \delta_B A^{(1)} = p\Lambda^{(1)}, \tag{123}$$

the term $\mathcal{F}^{(2)}$ is gauge invariant

$$\delta_B \mathcal{F}^{(2)} = p\delta_B B^{(2)} - \delta_B F^{(2)} = p\, d\Lambda^{(1)} - p\, d\Lambda^{(1)} = 0 \,, \tag{124}$$

and similarly for the variation with respect to $C^{(2)}$. The term on the (p, q) string transforms as

$$\int_{(p,q)\text{ string}} p\delta_B B^{(2)} - \int_{\partial(p,q)\text{ string}} \delta_B A^{(1)} = \int_{(p,q)\text{ string}} p\, d\Lambda^{(1)} - \int_{\partial(p,q)\text{ string}} p\Lambda^{(1)} = 0 \,, \tag{125}$$

which follows from Stokes' theorem. Hence, the action is now completely invariant under the variation of $B^{(2)}$ and similarly of $C^{(2)}$. It remains to check the what happens if we vary with respect to $A^{(1)}$, in this case we have

$$\delta_A A^{(1)} = d\phi^{(0)} \,, \tag{126}$$

with all other fields receiving zero variation. And the variation of the term on the boundary of the (p, q) string is

$$\int_{\partial(p,q)\text{ string}} \delta_A A^{(1)} = \int_{\partial(p,q)\text{ string}} d\phi^{(0)} = 0 \,, \tag{127}$$

again using Stokes' theorem.

To conclude, we have shown that the final gauge invariant action reproducing the equations of motions is

$$S = \int_{\mathbb{R}^{1,9}} H^{(3)} \wedge *_{10} H^{(3)} + G^{(3)} \wedge *_{10} G^{(3)} + \int_{(p,q)\text{ 5-brane}} \mathcal{F}^{(2)} \wedge *_6 \mathcal{F}^{(2)} \tag{128}$$

$$+ \int_{(p,q)\text{ string}} -pB^{(2)} - qC^{(2)} + \int_{\partial(p,q)\text{ string}} A^{(1)} \,. \tag{129}$$

6.7 D$(p-2)$ brane ending on Dp brane

Let us try to argue why we can have a D$(p-2)$ brane ending on a Dp brane. We already know that we can have a fundamental string F1 end on a Dp, and since the F1 brane is electrically charged under $B^{(2)}$, its boundary looks like

$$d *_{p+1} F^{(2)} = \delta^{(p)} \,. \tag{130}$$

And so we have a 1-form, the corresponding magnetically charged object looks like

$$dF^{(2)} = \delta^{(3)} \,, \tag{131}$$

which is a $p-3$ brane, or the boundary of a D$(p-2)$ brane.

6.8 Branes within branes

Furthermore, because we have a 2-form field strength, we can have a term like

$$\int_{\mathrm{D}p} C^{(p-3)} \wedge F^{(2)} \wedge F^{(2)} \tag{132}$$

???

7 Branes as Algebraic Objects

7.1 A_n algebra

There is a deep connection between brane systems and algebras. To illustrate this, we will use branes to construct an A_n algebra. Suppose we have n Dp branes, and to each we will assign an orthonormal basis element e_i with $i = 1, ..., n$ such that $(e_i, e_j) = \delta_{ij}$. Then there can be various F1 strings with different orientations between the different branes, see figure 7. At this point, we may also note that the moduli space here is

$$\mathcal{M} = \underbrace{\mathbb{R}^{9-p}}_{\text{Centre of Mass}} \times \frac{(\mathbb{R}^{9-p})^{n-1}}{S_n}\,, \tag{133}$$

and a string between two branes corresponds to a gauge boson with mass $m_{ij} \sim |\mathbf{x}_i - \mathbf{x}_j|$.

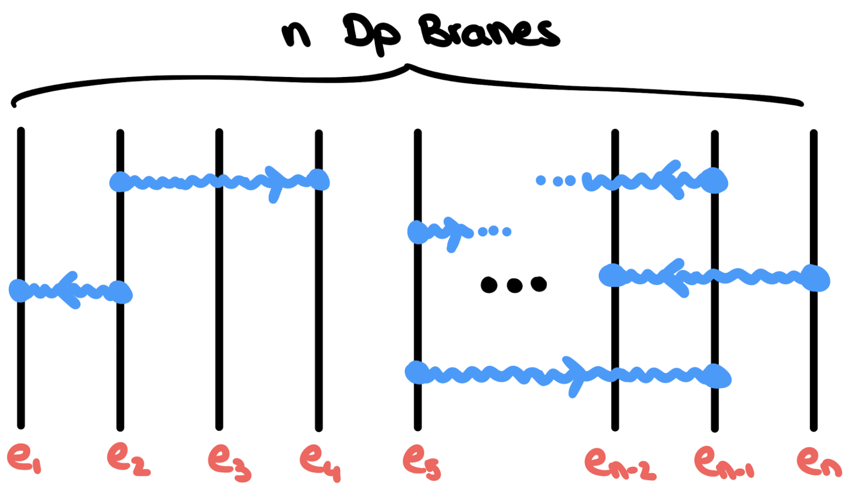


Figure 7: n Dp branes with various oriented strings between them.

Suppose now that we define a special set of these strings with a fixed orientation that connect neighbouring branes

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \dots \quad , \quad \alpha_{n-1} = e_{n-1} - e_n\,. \tag{134}$$

This can also be seen in figure 8. The strings of this type correspond to the simple roots of A_{n-1}.

To confirm that they indeed give us the A_{n-1} algebra, we first compute the inner products

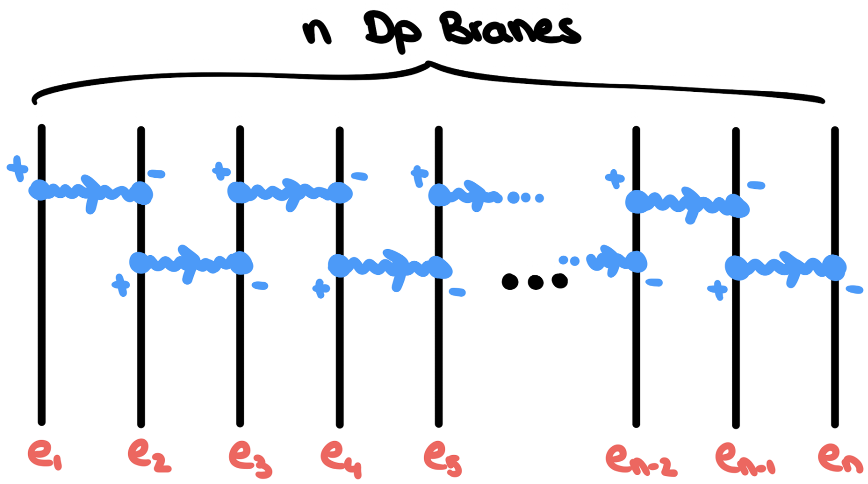


Figure 8: GET RID OF SIGNS. n Dp branes with strings, corresponding to simple roots of A_{n-1}, between them.

of the simple roots

$$(\alpha_i, \alpha_i) = 2 \tag{135}$$

$$(\alpha_i, \alpha_{i+1}) = -1 \tag{136}$$

$$(\alpha_i, \alpha_{j>i+1}) = 0\,. \tag{137}$$

And therefore, the Cartan matrix elements are

$$C_{i,i+1} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_i)} = -1 \tag{138}$$

$$C_{i,j>i+1} = 2\frac{(\alpha_i, \alpha_{j>i+1})}{(\alpha_i, \alpha_i)} = 0\,, \tag{139}$$

which gives us the correct Cartan matrix.

7.2 Orientifold planes

One can imagine, or guess by reading the section titles in the contents, that we can do something similar for the other classical algebras B_n, C_n and D_n. But before we can do so, we will need to introduce a new object into the mix.

The **orientifold plane** Op can be thought of as a reflection of spacetime in its transverse space $\mathbb{R}^{9-p}/\mathbb{Z}_2$. It is defined by combining the action of parity on the world sheet with a spacetime reflection. We summarise some of its properties, the Op plane

(i) is non-dynamical,

(ii) breaks the same amount of supersymmetry as a Dp brane,

(iii) reflects Dp branes from one side to the other, see figure 9,

(iv) is charged under a RR $(p+1)$-form,

(v) might have negative tension.

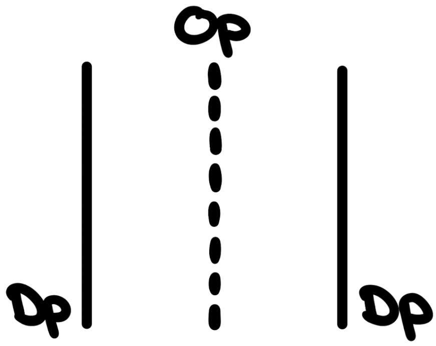


Figure 9: A Dp brane being reflected across an Op plane.

If we again put our n Dp branes back into this picture, then there are three general ways a string can connect on/across the orientifold plane. We begin with the so-called $\mathbf{\widetilde{O}p^-}$ **plane**. Here, a string is allowed to extend from one side of the orientifold plane across to the other side and connect to any brane except its image, but it is also allowed to end on the $\widetilde{O}p^-$ plane itself, see figure 10. We will also refer to this boundary as short, which will make more sense later on.

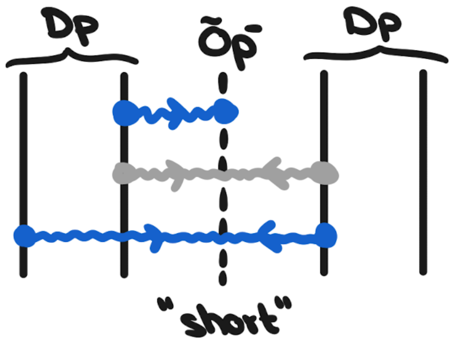


Figure 10: Boundary conditions of the $\widetilde{O}p^-$ plane.

Another possible boundary condition is the so-called $\mathbf{Op^+}$ **plane**. The string is no longer allowed to end on the Op^+ plane, but can now connect to its image, see figure 11. This is the long boundary condition.

The last possible configuration of strings is called the $\mathbf{Op^-}$ **plane**, which forbids both of the above cases, see figure 12. This could be called a non-boundary or split boundary.

As a last comment before we move on to constructing the other classical algebras, the charges of the different Op planes under the $C^{(p+1)}$ form are

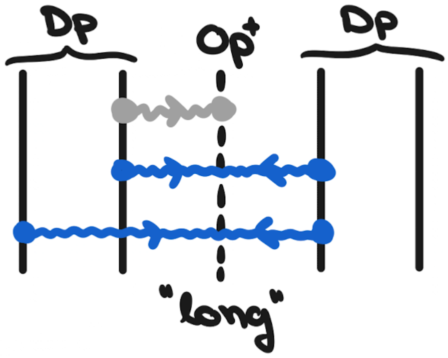


Figure 11: Boundary conditions of the Op^+ plane. The grey string is not allowed in this setup.

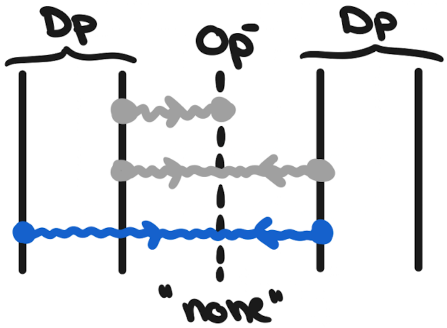


Figure 12: Boundary conditions of the Op^- plane. The grey strings are not allowed in this setup.

- $\widetilde{\text{O}}p^-$: $-2^{p-5}+\frac{1}{2}$
- Op^+: 2^{p-5}
- Op^-: -2^{p-5} .

Physically, the $\widetilde{O}p^-$ plane corresponds to an Op^- plane with half a Dp brane stuck on it. TENSIONS?

7.3 B_n, C_n and D_n algebras

$\mathbf{B_n}$ Algebra Starting with the B_n algebra, we can take n Dp branes and a $\widetilde{O}p^-$ plane. The setup is shown in figure 13, where we note that the branes on the opposite side of the orientifold have a negative basis element. We now choose the set of simple roots to be

$$\alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_n\,, \quad \alpha_n = e_n\,, \tag{140}$$

where the last string is between the rightmost brane and the orientifold plane, giving us a total of n simple roots. The positive roots are

$$e_i - e_j\,, \quad e_i - (-e_j) \quad \text{and} \quad e_i \quad \text{for} \quad j > i\,, \tag{141}$$

and, if you count them, you will find that there are n^2 of them, as is correctly the case for B_n. Since we also have strings with the opposite orientation, we naturally get the set of negative roots. Moving on to computing the Cartan matrix elements, the cases not involving α_n follow from our previous discussion on A_n, and the remaining cases are

$$C_{n-1,n} = 2\frac{(\alpha_{n-1}, \alpha_n)}{(\alpha_{n-1}, \alpha_{n-1})} = -1 \tag{142}$$

$$C_{n,n-1} = 2\frac{(\alpha_n, \alpha_{n-1})}{(\alpha_n, \alpha_n)} = -2\,, \tag{143}$$

which is, of course, what we needed.

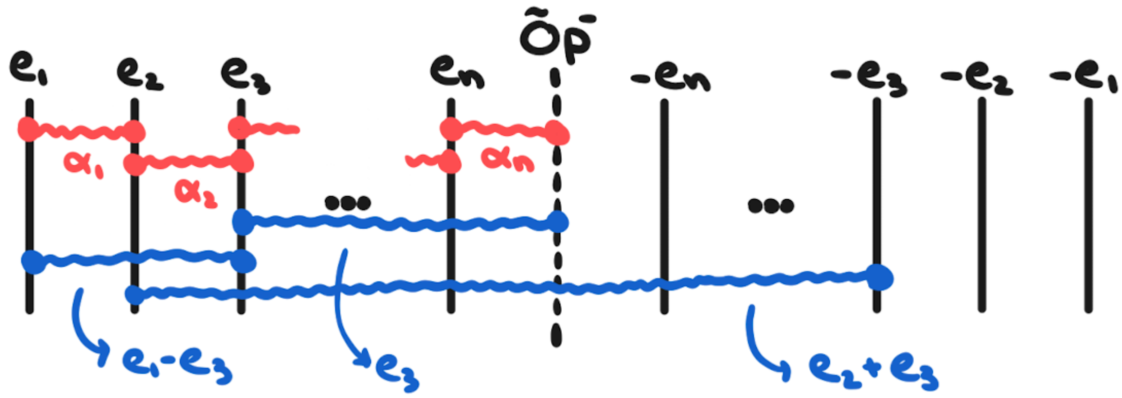


Figure 13: Brane and orientifold setup for the B_n algebra. Simple roots are shown in red, and some examples of other positive roots are shown in blue.

$\mathbf{C_n}$ Algebra Next up is the C_n algebra, we take n Dp branes as well as a Op^+ plane. The setup and our choice of simple roots are shown in figure 14. As one can see from the figure, the simple roots are

$$\alpha_1 = e_1 - e_2, \quad \dots, \quad \alpha_{n-1} = e_{n-1} - e_n\,, \quad \alpha_n = 2e_n\,, \tag{144}$$

and the other positive roots

$$e_i - e_j \quad \text{for} \quad i < j\,, \quad e_i - (-e_j) \quad \text{and} \quad 2e_i \quad \text{for any} \quad i, j\,. \tag{145}$$

Again, it is easy to confirm that this replicates the Cartan matrix elements correctly

$$C_{n-1,n} = 2\frac{(\alpha_{n-1}, \alpha_n)}{(\alpha_{n-1}, \alpha_{n-1})} = -2 \tag{146}$$

$$C_{n,n-1} = 2\frac{(\alpha_n, \alpha_{n-1})}{(\alpha_n, \alpha_n)} = -1\,. \tag{147}$$

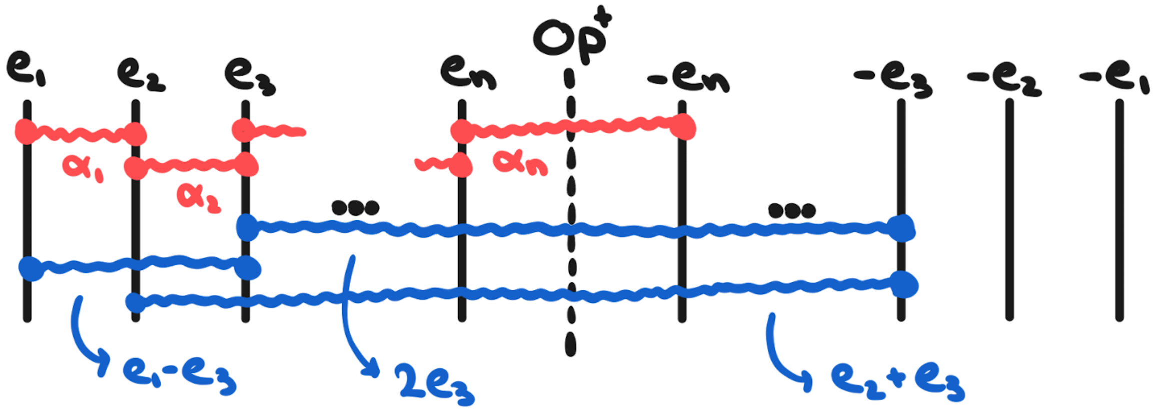


Figure 14: Brane and orientifold setup for the C_n algebra. Simple roots are shown in red, and some examples of other positive roots are shown in blue.

$\mathbf{D_n}$ Algebra For the last case, we need n Dp branes and the last remaining Op^- plane. We choose the simple roots to be

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n\,, \quad \alpha_n = e_{n-1} + e_n\,, \tag{148}$$

as can also be seen in figure 15, and the positive roots are

$$e_i - e_j \quad \text{for} \quad i < j\,, \quad e_i - (-e_j) \quad \text{for any} \quad i, j\,. \tag{149}$$

The relevant Cartan matrix elements are

$$C_{n-2,n} = 2\frac{(\alpha_{n-2}, \alpha_n)}{(\alpha_{n-2}, \alpha_{n-2})} = -1 \tag{150}$$

$$C_{n,n-2} = 2\frac{(\alpha_n, \alpha_{n-2})}{(\alpha_n, \alpha_n)} = -1 \tag{151}$$

$$C_{n-1,n-2} = 2\frac{(\alpha_{n-1}, \alpha_{n-2})}{(\alpha_{n-1}, \alpha_{n-1})} = 0 \tag{152}$$

$$C_{n-2,n-1} = 2\frac{(\alpha_{n-2}, \alpha_{n-1})}{(\alpha_{n-2}, \alpha_{n-2})} = 0\,. \tag{153}$$

7.4 Adjoint Higgs mechanism

We know that a single Dp brane supports a $U(1)$ gauge field, and if we have n separate branes, this leads to a $U(1)^n$ gauge symmetry. We also know that a string between two branes corresponds to a massive gauge boson with mass $m_{ij} \sim |\mathbf{x}_i - \mathbf{x}_j|$. The indices $i, j = 1, \ldots, n$ are also referred to as **Chan-Paton indices**. But this implies that, if all branes coincide, we have a total of n^2 vector multiplets and the gauge group enhances to the adjoint of $U(n)$. This is called the **adjoint Higgs mechanism**, because analogous to the Higgs mechanism, moving the branes apart gives masses to gauge fields.

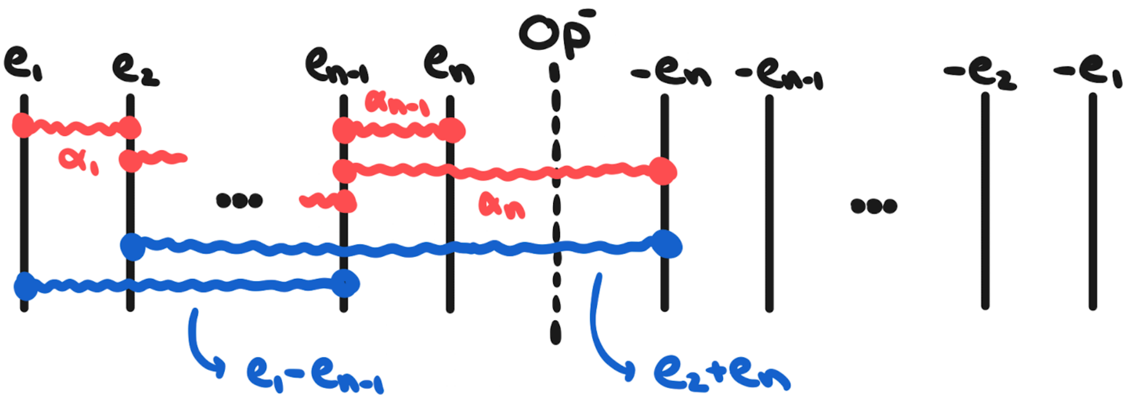


Figure 15: Brane and orientifold setup for the D_n algebra. Simple roots are shown in red, and some examples of other positive roots are shown in blue.

For the case of n Dp branes, suppose we have multiple groups of coinciding branes, for example 4 Dp branes coincide at position $\mathbf{x}_1$ and 20 Dp branes coincide at position $\mathbf{x}_2$. Then we label the number of coinciding branes by n_i, and this will form a partition of the total number of branes $n = n_1 + \cdots + n_k$. The most general form of the gauge group is

$$\prod_{i=1}^{k} U(n_i)\,, \tag{154}$$

where $\{n_i\}$ is the said partition.

This now generalises straightforwardly using our earlier results on how to construct different gauge groups with orientifold planes. We will now use n_0 to denote the number of branes coinciding on the orientifold itself, and $n_1, \ldots, n_k$ for other groupings. The result for the different Op planes is

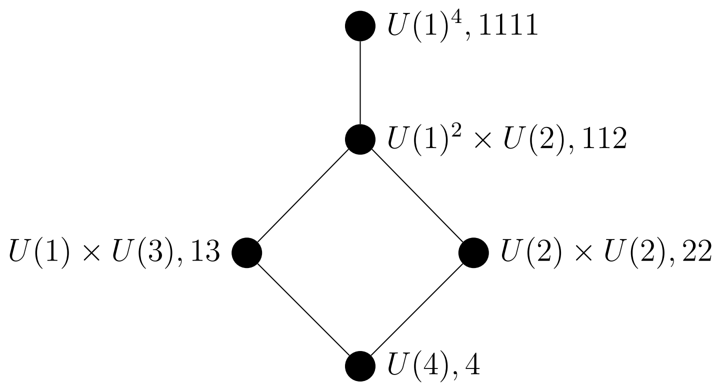
$$\widetilde{O}p^- : \quad O(2n_0+1) \times \prod_{i=1}^{k} U(n_i) \tag{155}$$

$$Op^+ : \quad Sp(n_0) \times \prod_{i=1}^{k} U(n_i) \tag{156}$$

$$Op^- : \quad O(2n_0) \times \prod_{i=1}^{k} U(n_i)\,, \tag{157}$$

where of course the B_n algebra corresponds to the group $O(2n+1)$, the C_n algebra to the group $Sp(n)$, and the D_n algebra to $O(2n)$. All these groups are called the **Levi subgroups**.

This singularity structure can also be encoded in a **Hasse diagram**, for example, in the case of four Dp branes



We have not discussed two important points. One of them is that, instead of strings between Dp branes, we could equally well have D$(p-2)$ branes between Dp branes. This will give the magnetic spectrum and the boundary conditions of the Op planes changes. The other point which we have neglected is the existence of a fourth orientifold plane, the $\widetilde{\mathbf{O}}\mathbf{p}^+$ **plane**. The complete result may now be summarised as in table 5.

Op Plane	F1	D$(p-2)$
Op^-	None (D_n)	None (D_n)
$\widetilde{O}p^-$	Short (B_n)	Long (C_n)
Op^+	Long (C_n)	Short (B_n)
$\widetilde{O}p^+$	Long (C_n)	Long (C_n)

Table 5: Boundary Conditions for different Op planes and the resulting algebras.

7.5 Type I

We can now finally talk about the 10 dimensional theory which we have not discussed in section 4. **Type I** is a theory of open strings and unoriented closed strings, so of Neumann boundary conditions only. We also recall that for an F1 brane ending on a Dp brane, it is subject to $p+1$ Neumann boundary conditions and $9-p$ Dirichlet boundary condition. And therefore, a D9 brane extending through all of spacetime would give us just the right conditions.

A problem arises if look at how the D9 brane couples to the 10-form $A^{(10)}$ with field strength $F^{(11)}$

$$d *_{10} F^{(11)} = Q\delta^{(0)} . \tag{158}$$

The corresponding term in the action can be written as

$$Q \int_{D9} A^{(10)} , \tag{159}$$

which is effectively a Lagrange multiplier that forces $Q = 0$ on-shell. To make the two statements above agree with each other, one that the charge is $Q = 1$ as in (158), and two that

$Q = 0$ as in (159), we note that O9$^-$ has charge -16. If we hence stack 16 D9 branes on top of an O9$^-$ plane, we get an object with overall charge $Q = 0$, yet each of the D9 branes can still couple to the 10-form with charge $Q = 1$. It follows that the gauge symmetry is $O(32)$ and since branes are half-BPS states, there are 16 supercharges.

8 Aside: Classical Solutions

Tong TASI lectures

8.1 Gauge Instantons

codim = 4

D5 inside D9, F1 inside NS5, Dp inside D(p+4)

8.2 Monopoles

codim = 3

D6, D3 on NS5, Dp ending on D(p+2)

't Hooft Polyakov monopole

8.3 Vortices

codim = 2

D7, D4 ending on NS5, F1 ending on D2

logarithmic behavious, deficit angle

8.4 Domain Walls

codim = 1

D8, D5 ending on NS5, F1 ending on D1

9 Dualities

9.1 Parameters in string theory

In all 10-dimensional theories, we have a dilaton Φ in the universal NS-NS sector that couples to the worldsheet as

$$\int d^2\sigma\,\sqrt{h}R\Phi\,, \tag{160}$$

where h is the determinant of the worldsheet metric and R is the Ricci scalar encoding the curvature of the worldsheet. And if Φ admits a vacuum expectation value Φ_0, we have the following neat result

$$\begin{aligned}\frac{\Phi_0}{4\pi}\int d^2\sigma\,\sqrt{h}R &= \Phi_0\chi \\ &= \Phi_0(2-2g-h)\end{aligned} \tag{161}$$

where we have (for once) included the necessary constants, g is the genus or number of holes, and h is the number of boundaries.

To illustrate this, we can consider a scattering process in the path integral formulation

$$e^{-S} = e^{-\Phi_0(2-2g-h)}\,. \tag{162}$$

Suppose we have some 2-2 scattering, as shown in figure 16, then we can compute the ratio of the one-loop correction to the tree-level contribution to be $e^{2\Phi_0}$. And the **string coupling** is

$$g_s = e^{\Phi_0}\,. \tag{163}$$

In the case of small $g_s << 1$, we can use perturbative methods in calculations, but if $g_s \sim 1$, we cannot use perturbative methods.

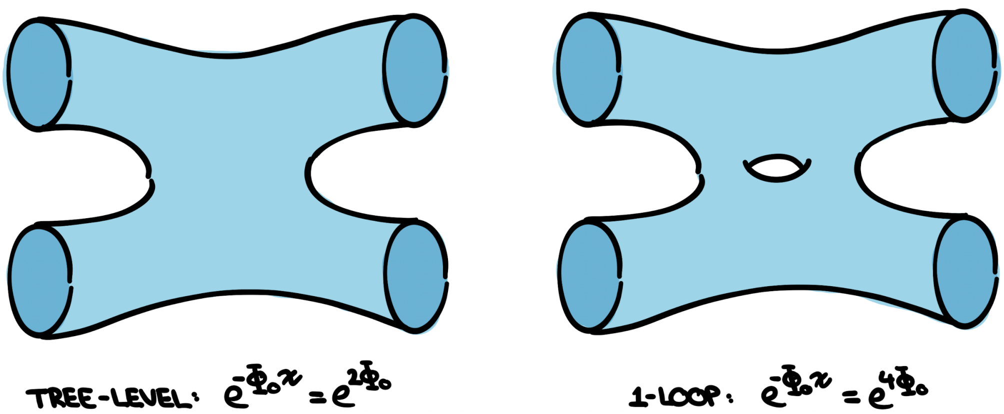


Figure 16: Scattering of a string.

When we look at the kinetic term in the action of a string

$$T_{\mathrm{F1}} \int d^2\sigma \sqrt{h} h^{ij} \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}\,, \tag{164}$$

where

$$T_{\mathrm{F1}} = \frac{1}{l_s^2}\,, \tag{165}$$

is the **tension of the F1 brane**. The parameters g_s and l_s (or T) completely specify the objects in our theories. For Dp branes, we can have an F1 ending on them, which means that the Euler number of this setup is $\chi = 1$ due to the boundary. This contributes a factor of $e^{-S} = e^{-\Phi_0} = 1/g_s$ and hence the **tension of a Dp brane** is

$$T_{\mathrm{D}p} = \frac{1}{g_s l_s^{p+1}}\,. \tag{166}$$

Similarly, for the NS5 brane the Euler number is $\chi = 0$ and the **tension of the NS5 brane** becomes

$$T_{\mathrm{NS5}} = \frac{1}{g_s^2 l s^6}\,. \tag{167}$$

In M-theory, we do not have a microscopic description. But for 11d supergravity, we do have a metric and so can write the Einstein-Hilbert action

$$\frac{1}{G} \int d^{11}x \sqrt{g} R\,. \tag{168}$$

From this, we can tell that G needs to have the following length scale

$$[G] = l_p^9\,, \tag{169}$$

where we have defined the Planck length as the fundamental length scale. The **tension of the M2 brane** becomes

$$T_{\mathrm{M2}} = \frac{1}{l_p^3}\,, \tag{170}$$

and the **tension of the M5 brane** is

$$T_{\mathrm{M5}} = \frac{1}{l_p^6}\,. \tag{171}$$

9.2 Dualities for theories with 32 supercharges

9.2.1 M theory on S^1 $\leftrightarrow$ Type IIA

We begin with a duality which we have already encountered, this is the duality resulting from compactifying M theory to get type IIA, we say that: M theory on $S^1 \times \mathcal{M}_{10}$ is dual to type IIA on $\mathcal{M}_{10}$. Our task is essentially to match the parameters of M theory (the Planck length l_p and the radius of the cirlce R) to the parameters of type IIA (the dilaton g_s and the fundamental string length l_s).

Since there are two parameters, we need to postulate two duality relations and the remaining ones can be calculated from our result. From considering the dimensions we have that

$$\text{M2 on } S^1 \quad \leftrightarrow \quad \text{F1} \tag{172}$$

$$\text{M2} \quad \leftrightarrow \quad \text{D2}\,. \tag{173}$$

Matching the tensions of these objects,

$$T_{\text{M2 on } S^1} = T_{\text{F1}} \qquad \Longrightarrow \qquad \frac{R}{l_p^3} = \frac{1}{l_s^2} \tag{174}$$

$$T_{\text{M2}} = T_{\text{D2}} \qquad \Longrightarrow \qquad \frac{1}{l_p^3} = \frac{1}{g_s l_s^3}\,, \tag{175}$$

and therefore the equations characterising the duality are

$$l_p^3 = g_s l_s^3 \quad \text{and} \quad R = g_s l_s\,. \tag{176}$$

Using this result, we can derive the other duality relations. The complete list is then

$$\text{M2 on } S^1 \quad \leftrightarrow \quad \text{F1} \tag{177}$$

$$\text{M2} \quad \leftrightarrow \quad \text{D2} \tag{178}$$

$$\text{M5 on } S^1 \quad \leftrightarrow \quad \text{D4} \tag{179}$$

$$\text{M5} \quad \leftrightarrow \quad \text{NS5} \tag{180}$$

$$\text{Momentum Mode} \quad \leftrightarrow \quad \text{D0} \tag{181}$$

$$\text{Kaluza-Klein Monopole} \quad \leftrightarrow \quad \text{D6}\,, \tag{182}$$

where we have used the inverse relations of (176) to find that the D0 brane is dual to the so-called **momentum mode** (the excitation mode from wrapping a string around a circle) and that the D6 brane is dual to the **Kaluza-Klein monopole** (a monopole carrying the magnetic charge of a $U(1)$ Kaluza-Klein gauge symmetry).

From (176) we can tell that for the case of large g_s, we are essentially growing another dimension, and can consider type IIA as M theory. Since the strong coupling limit is related to a different theory we call this duality an **S-duality**.

9.2.2 Type IIB $\leftrightarrow$ Type IIB: The $SL(2,\mathbb{Z})$ Duality

Suppose we have two type IIB theories with different parameters: g_s, l_s and g'_s, l'_s. They are related by a duality where

$$\text{F1} \quad \leftrightarrow \quad \text{D1} \tag{183}$$

$$\text{D3} \quad \leftrightarrow \quad \text{D3}\,, \tag{184}$$

which gives us the relations

$$l_s^2 = g'_s l'^2_s \quad \text{and} \quad g_s = \frac{1}{g'_s}\,. \tag{185}$$

And from this, we can also find that

$$\text{D5} \quad \leftrightarrow \quad \text{NS5} \tag{186}$$

$$\text{D7} \quad \leftrightarrow \quad \text{Vortex}\,, \tag{187}$$

where the vortex is a codimension 2 object. We can also see that we again have an S-duality as the strong and weak coupling are interchanged.

So far, we have disregarded that we not only have a dilaton, but also an axion. And recall that the moduli space of type IIB is

$$\mathcal{M} = \frac{SL(2,\mathbb{R})}{SO(2)}\,. \tag{188}$$

This, as it turns out, gives the geometry of a torus. We shall look at exactly what we mean by this before returning to the duality.

Aside: Geometry of a torus

Consider an unfolded torus, with sides identified, see figure 17. We can now define two parameters

$$\tau = i\frac{R_1}{R_2} + \frac{\theta}{2\pi} \quad \text{and} \quad A = R_1 R_2\,, \tag{189}$$

encoding its shape and size, respectively. The torus is invariant under **modular transformations**, which are transformations of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})\,. \tag{190}$$

These transformations are generated by

$$\tau \to -\frac{1}{\tau} \quad \text{and} \quad \tau \to \tau + 1\,, \tag{191}$$

where we will shortly see their meaning in the context of type IIB.

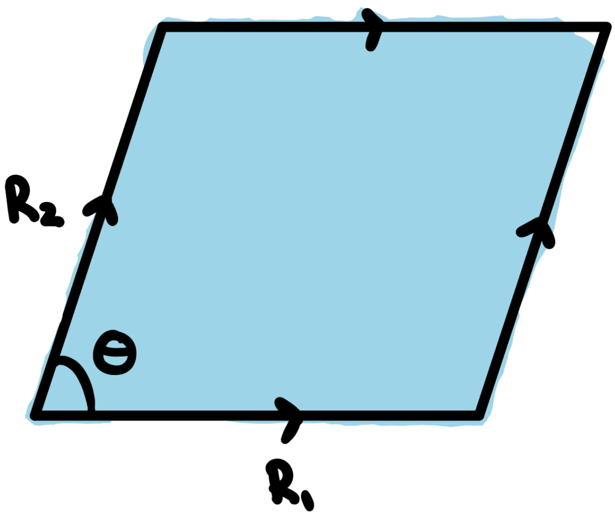


Figure 17: Torus parameters.

$SL(2,\mathbb{Z})$ for type IIB

This can now be applied to type IIB. We define

$$\tau = i\frac{1}{g_s} + \frac{C^{(0)}}{2\pi}\,, \tag{192}$$

where $\frac{1}{g_s}$ corrsponds to the dilaton and $C^{(0)}$ is the axion. Then (191) translates into

$$\underbrace{\tau = i\frac{1}{g_s} \quad \mapsto \quad -\frac{1}{\tau} = -\frac{g_s}{i} = ig_s}_{\text{S-duality}}\,, \tag{193}$$

which is what we had already found, and

$$\tau = i\frac{1}{g_s} + \frac{C^{(0)}}{2\pi} \quad \mapsto \quad \tau + 1 = i\frac{1}{g_s} + \frac{C^{(0)}}{2\pi} + 1 = i\frac{1}{g_s} + \frac{C^{(0)} + 2\pi}{2\pi}\,, \tag{194}$$

which is just saying that $C^{(0)} = C^{(0)} + 2\pi$, as we already know since $C^{(0)} \in S^1$.

We may also incorporate (p,q) strings and branes into our analysis, for these objects, the tension can be written as

$$T_{(p,q)\text{ string}} = \frac{1}{l_s^2}|p + q\tau| \tag{195}$$

$$T_{(p,q)\text{ 5-brane}} = \frac{1}{g_s l_s^2}|p + q\tau|\,. \tag{196}$$

9.2.3 Type IIA on S^1 $\leftrightarrow$ Type IIB on S^1

Since type IIA and type IIB reduce to the same theory in 9-dimensions, there will be a duality between the two theories on different circles. We have the parametrs: g_A, l_A, R_A in type IIA and g_B, l_B, R_B in type IIB. As we have 3 terms, we will also need 3 equations defining the duality. On dimensional grounds, we know that a Dp brane in one theory on a small circle becomes a D$(p-1)$ brane in the other theory. Let us consider the following

$$\text{F1} \quad \leftrightarrow \quad \text{F1} \tag{197}$$

$$\text{D2 on } S^1 \quad \leftrightarrow \quad \text{D1} \tag{198}$$

$$\text{D2} \quad \leftrightarrow \quad \text{D3 on } S^1\,. \tag{199}$$

From (197), we can immediately tell that

$$l_A^2 = l_B^2\,, \tag{200}$$

which we hence just call l_s^2. The other equations, (198) and (199), give us

$$\frac{R_A}{g_A l_s^3} = \frac{1}{g_B l_s^2} \quad \text{and} \quad \frac{1}{g_A l_s^3} = \frac{R_B}{g_B l_s^4}\,, \tag{201}$$

after a little rearranging, they read

$$R_A R_B = l_s^2 \quad \text{and} \quad \frac{R_A}{g_A^2} = \frac{R_B}{g_B^2}\,. \tag{202}$$

Historically, T-duality was first seen to exchange the momentum mode with the winding mode of an F1 string on a circle, i.e.

$$\text{Winding Mode}\left(T = \frac{R_A}{l_s^2}\right) \quad \leftrightarrow \quad \text{Momentum Mode}\left(T = \frac{1}{R_B}\right). \tag{203}$$

The mass of this string is given by

$$m_{(w,n)} = \frac{wR}{l_s^2} + \frac{|n|}{R}\,, \tag{204}$$

and T-duality exchanges $(w, n) \leftrightarrow (n, w)$.

9.2.4 M theory on T^2 $\leftrightarrow$ Type IIB on S^1

By the same reason that we have a duality between type IIA on S^1 and type IIB on S^1 (due to both theories reducing to the same 9 dimensional theory), we will also have a duality between M-theory on T^2 and type IIB on S^1, as well as type IIA for that matter.

As we are still in 9 dimensions, we have three parameters. For type IIB, they are the same as before l_B, g_B and R_B. For M theory they are l_p, R_1 and R_2, where R_1 and R_2 correspond to the two radii of the torus.

Doing a step-by-step analysis, going from M theory to type IIA via S-duality and from type IIA to type IIB via T-duality (see figure 18 and 19), we can see what sort of branes are dual

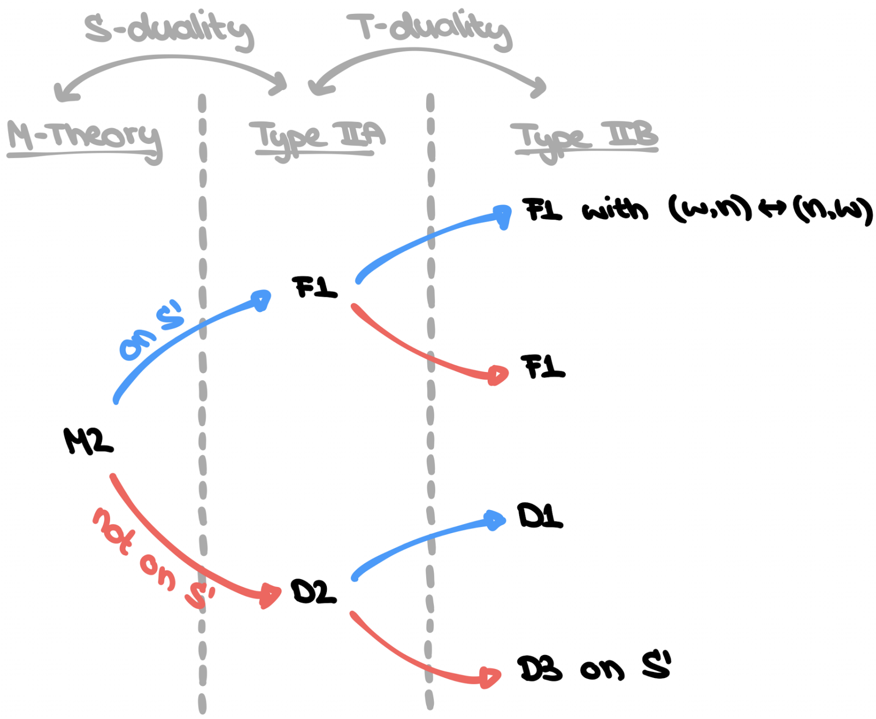


Figure 18: M2 brane to different object in type IIB.

to each other in M theory on T^2 and type IIB on S^1. From this, we choose 3 dualities (for 3 parameters), say,

$$\text{M2} \quad \leftrightarrow \quad \text{D3 on } S^1 \tag{205}$$
$$\text{M2 on } S^1_{R_2} \quad \leftrightarrow \quad \text{D1} \tag{206}$$
$$\text{M2 on } S^1_{R_1} \quad \leftrightarrow \quad \text{F1}\,. \tag{207}$$

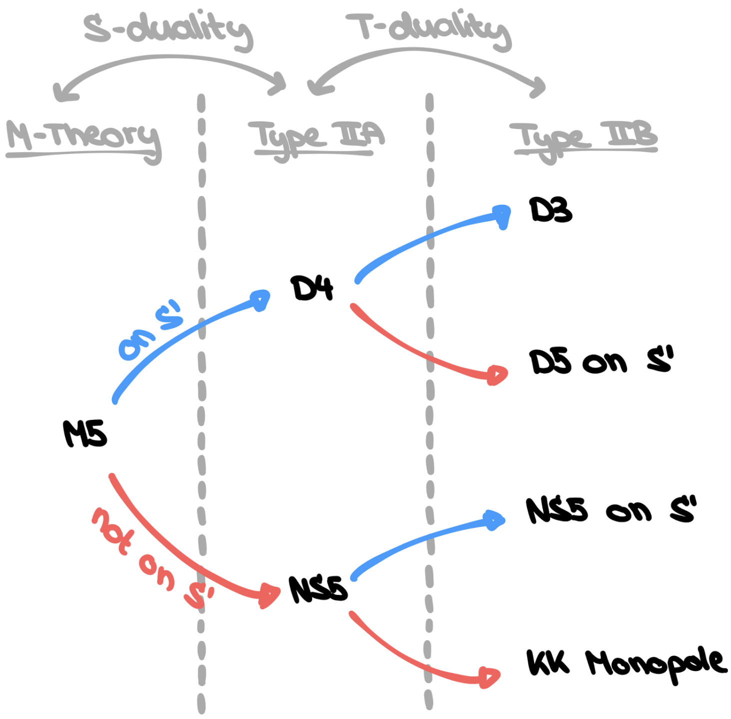


Figure 19: M5 brane to different object in type IIB.

One can then straightforwardly compute the relations to be

$$l_p^3 = g_B \frac{l_B^4}{R_B}\,, \quad R_1 = g_B \frac{l_B^2}{R_B} \quad \text{and} \quad R_2 = \frac{l_B^2}{R_B}\,. \tag{208}$$

If we want to extend our analysis to (p, q) strings, we need to consider the different ways of compactifying on a torus. These are called (p, q) cycles and can be seen in figure 20. If p and q are not coprime, we have that, for example, $(3, 9) = 3 \times (1, 3)$. We begin by looking at the two

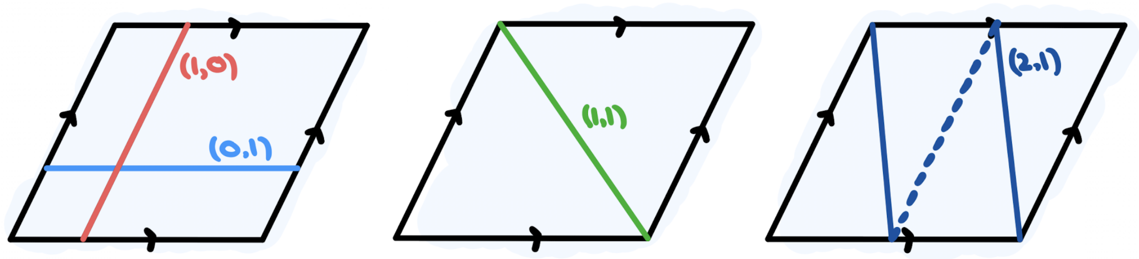


Figure 20: Different (p, q) cycles.

base cases of M2 becoming an F1 or D1 brane, depending on which circle it is compactified on. The tension of a general (p, q) string in type IIB may be written as

$$T_{(p,q)\ \text{string}} = \frac{1}{l_B^2}|p + q\tau| = \frac{1}{l_B^2}\sqrt{p^2 + \frac{q^2}{g_B^2}}\,, \tag{209}$$

and using the inverse equations of (208) gives us

$$T_{(p,q)\text{ string}} = \frac{R_1}{l_p^3}\sqrt{p^2 + q^2\frac{R_2^2}{R_1^2}} \tag{210}$$

$$= \frac{1}{l_p^3}\sqrt{p^2R_1^2 + q^2R_2^2} \tag{211}$$

$$= T_{\text{M2 on }(p,q)\text{ cycle}}\,, \tag{212}$$

where in the last line we recognise this as the tension of an M2 brane on a (p, q) cycle.

There is another interesting feature we can look at if we consider M2 wrapping both circles, giving us the momentum mode of type IIB. The tensions for this case reads

$$\frac{A}{l_p^3} = \frac{1}{R_B}\,, \tag{213}$$

where we defined $A = R_1R_2$ (the size modulus of the torus). This equation is telling us that, in the limit of the torus becoming smaller, normal type IIB (not on S^1) is equivalent to M theory on a very small torus.

9.2.5 Overview

We can view the various theories in this section as being different limits of M theory on T^2, see figure 21. In the case of $R_1, R_2 \to \infty$, we have genuine M theory/11d supergravity. If one of

images/Mtheory_limits.png

Figure 21: Various limits of M theory on T^2.

the radii goes to zero, $R_1 \to 0$, we simply get type IIA theory by dimensional reduction. And, lastly, when both radii go to zero $R_1, R_2 \to 0$ we get type IIB. But, depending on which edge we transverse, we either get the duality relation

$$g_B = \frac{R_1}{R_2} \quad \text{or} \quad g'_B = \frac{R_2}{R_1}\,, \tag{214}$$

which is, of course, nothing but the type IIB self-duality statement $g_B = 1/g'_B$.

9.3 Dualities for theories with 16 supercharges

9.3.1 Type I $\leftrightarrow$ Heterotic $SO(32)$

Recall that type I is equivalent to type IIB with an $O9^-$ plane and 16 D9 branes stacked on top of it. In type I, we have a D1 and a D5 brane, in heterotic $SO(32)$, we do not have an RR sector, and therefore the only branes we have are F1 and NS5. It follows that

$$\text{D5} \quad \leftrightarrow \quad \text{NS5} \tag{215}$$

$$\text{D1} \quad \leftrightarrow \quad \text{F1}\,, \tag{216}$$

and the equations relating the two theories are thus

$$g_I = \frac{1}{g_H} \quad \text{and} \quad l_I^2 = g_H l_H^2\,, \tag{217}$$

which is akin to the self-duality of type IIB.

9.3.2 Heterotic $SO(32)$ on S^1 $\leftrightarrow$ Heterotic $E_8 \times E_8$ on S^1

Similar to the T-duality between type IIA and type IIB theories, we have a duality between the two heterotic theories. The duality is characterised by

$$R_{SO(32)} R_{E_8\times E_8} = l_s^2 \quad \text{and} \quad \frac{R_{SO(32)}}{g^2_{SO(32)}} = \frac{R_{E_8\times E_8}}{g^2_{E_8\times E_8}}\,. \tag{218}$$

9.3.3 Type I on S^1 $\leftrightarrow$ Type I'

As we have seen plenty of times by now, type I can be constructed by placing an $O9^-$ plane and 16 D9 branes in type IIB. If we now compactify on a circle, we know that there is a T-duality between type IIA and type IIB, and therefore there is a duality between type I on S^1 and type IIA on $S^1/\mathbb{Z}_2 \simeq I$, which is usually referred to as type I'. This is best seen in figure 22.

9.3.4 Overview

9.4 Note on additional theories with 16 supercharges

9.5 Summary of dualities

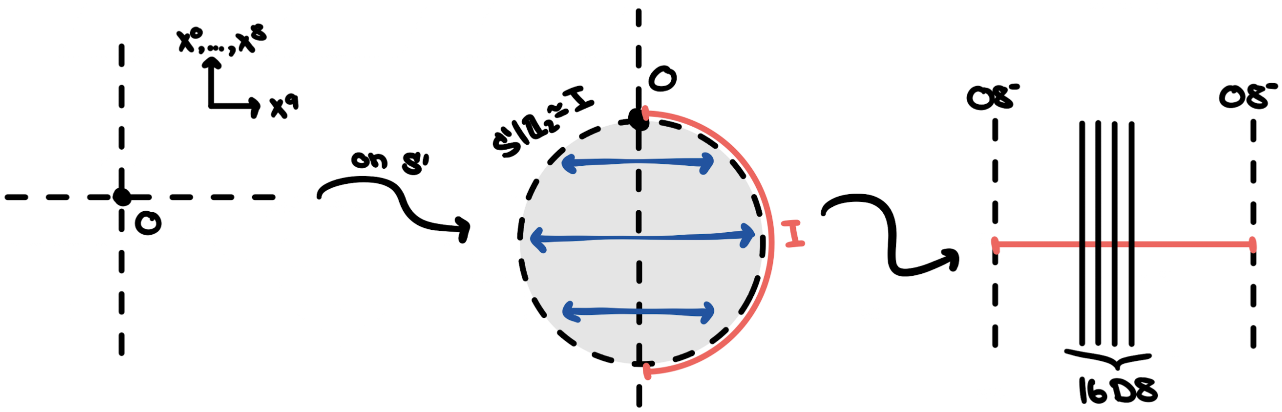


Figure 22: Construction of type I' by compactifying a spatial direction of type I. In the last picture on the right we have added the 16 D8 branes back in.

10 Theories with 8 Supercharges

Since branes break half the supersymmetry, we want to consider constructions with branes that give us 8 supercharges, i.e. 1/4 of the initial 32 supercharges. There are 2 brane systems which accomplish that, as seen in figure 23.

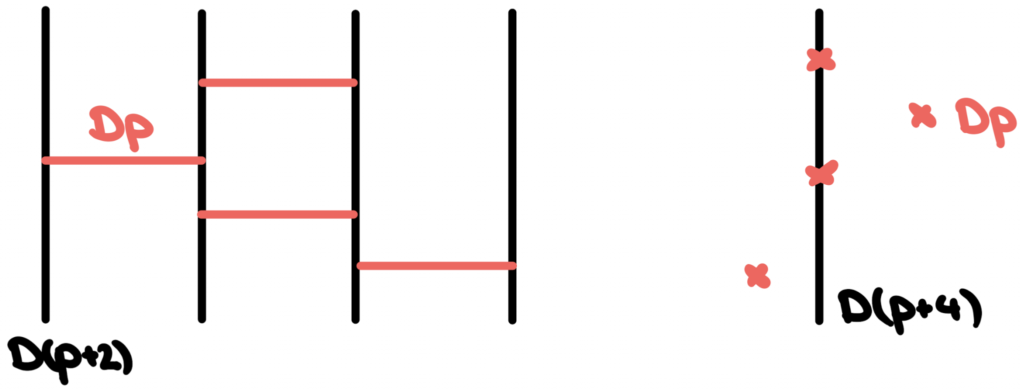


Figure 23: Brane constructions for theories with 8 supercharges.

10.1 D5-D9 brane system

As a concrete example, let us take a D5 brane inside a D9 brane. The D5 brane breaks the spacetime into $\mathbb{R}^{1,5} \times \mathbb{R}^4 \subset \mathbb{R}^{1,9}$ thus we will have a $SO(4) \simeq SU(2) \times SU(2)$ symmetry. This is not a full R-symmetry like we had in the case of theories with 16 supercharges. Recall that for 6-dimensional theories with (p, q) supersymmetry, and $8(p+q)$ supercharges, the R-symmetry is given by

$$Sp(p) \times Sp(q)\,. \tag{219}$$

Because we only have 8 supercharges on the world volume of the D5 brane, our R-symmetry is $Sp(1) \simeq SU(2)$. Going back to the symmetry of the $\mathbb{R}^4$ space, we therefore have

$$SO(4) \simeq \underbrace{SU(2)}_{\text{R-symmetry}} \times \underbrace{SU(2)}_{\text{Global Symmetry}}\,. \tag{220}$$

Let us also investigate the type of multiplets we can have. For 8 supercharges, we want a multiplet with $2^{8/4} = 4$ degrees of freedom transforming under the little group $SO(4)$ and R-symmetry $SU(2)$. Since bosonic and fermionic degrees of freedom still have to match we have the so-called **half-hyper multiplet**

$$h = [00]_4[1]_R + [01]_4[0]_R\,. \tag{221}$$

From this, we can construct various other multiplets, such as the vector, tensor and gravity multiplets:

$$\text{vplet:} \quad h[10][0] = [11][0] + [10][1] \tag{222}$$
$$\text{tplet:} \quad h[01][0] = [02][0] + [00][0] + [01][1] \tag{223}$$
$$\text{gplet:} \quad h[21][0] = [22][0] + [02][0] + [21][1]\,. \tag{224}$$

On the D9 brane, we initially have a vector multiplet V_{10} but due to the presence of the D5 brane this will decompose as

$$V_{10} = [1000]_8 + [0001]_8 \quad \rightarrow \quad \underbrace{[11][00] + [01][01]}_{\text{vplet}} + \underbrace{[00][11] + [10][10]}_{\text{hplet } = 2h} \tag{225}$$
$$= \quad V_6 + H_6 \tag{226}$$

since the spacetime symmetry is now $SO(4) \times SO(4) \simeq SO(4) \times SU(2)_R \times SU(2)_G \subset SO(8)$.

For the spacetime perspective we can draw a picture like in figure 24. Since the D5 branes are codimension 4 objects and since we are in type I, they are instantons with gauge group $SO(32)$. All the possible configurations of the D5 branes make up the moduli space of n $SO(32)$ instantons on $\mathbb{R}^4$. On the D5 brane, we have a novel feature, instead of having a $U(1)$ gauge

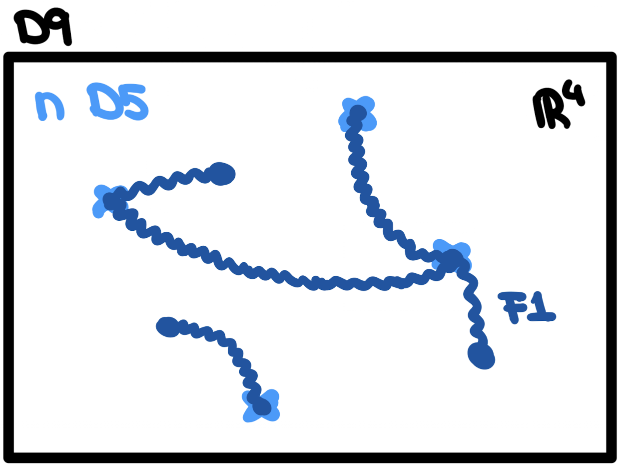
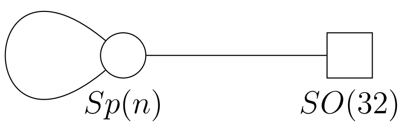


Figure 24: Spacetime perspective of n D5 branes inside D9 brane.

symmetry, the presence of the $O9^-$ plane induces a $Sp(1)$ gauge symmetry. And for n D5 branes coinciding, we have a $Sp(n) \subset Sp(1)^n$ gauge symmetry. We can summarise this information in a **quiver diagram**



where the loop stands for strings in the adjoint representation of $Sp(n)$, $\Lambda^2[10...0]_{Sp(n)}$, and the edge stands for the strings in the bifundamental representation of $Sp(n) \times SO(32)$, i.e. $[10...0]_{Sp(n)}[10...0]_{SO(32)}$. We can also have strings from the D9 brane to the D9 brane, they will correspond to parameters on the world volume of the Dp brane.

10.2 Dp-D$(p+4)$ brane system

We consider the Dp-D$(p+4)$ brane system, of which our previous case D5-D9 is a special case. But in many regards, the limiting $-1 \leq p \leq 4$ in the analysis makes things easier. Since the heavier brane does not occupy all spacetime directions, we can have transverse and parallel directions with the lighter brane, schematically this is shown in figure 25. As we have already

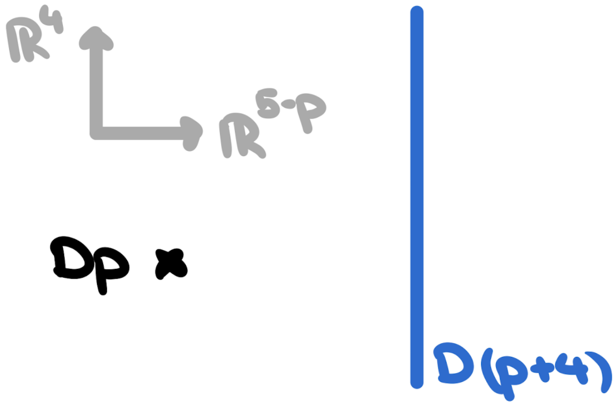


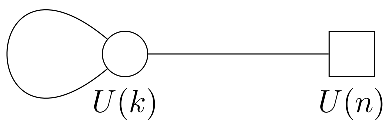
Figure 25: Dp-D$(p+4)$ brane system where we have 4 parallel directions and $(5-p)$ transverse directions.

seen, the number of scalars in the different multiplets are

$$H_d: \quad 4 \tag{227}$$
$$V_d: \quad 6-d \tag{228}$$
$$T_d: \quad 1\,. \tag{229}$$

This means that the hyper multiplet correctly encodes the degrees of freedom from the directions parallel to the heavier brane and the vector multiplet encodes the once transverse to it.

For k Dp branes next to n D$(p+4)$ brane, we again write a quiver:



From the perspective of the Dp brane, $U(n)$ is a global symmetry, but from the perspective of the D$(p+4)$ brane it is a gauge symmetry.

When the lighter brane is away from the heavier brane, the scalars in the vector multiplet, which parameterise the distance, form a moduli space called the **Coulomb branch**. The number of scalars is given by $k-(6-d)$ for different dimensions. The scalars in the hypermultiplet have mass which is given by the distance of the brane. But if the lighter brane moves inside the heavier brane, the degrees of freedom in the hypermultiplet become massless and can admit a vacuum expectation value, whereas the scalars in the vector multiplet are fixed to zero. The moduli space of the scalars in the hypermultiplet is called the **Higgs branch**.

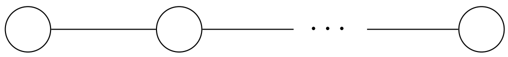
Furthermore, if the Dp branes are inside the D$(p+4)$ branes, the Dp branes look like instantons. This tells us that the Higgs branch is the same as the moduli space of k $U(n)$ instantons on $\mathbb{R}^4$.

10.3 D3-NS5 brane system

A Derivation of Dimension Formulas

A.1 A_n

Recall that $A_n \simeq su(n+1)$ with Dynkin diagram



and its dimension and number of roots are given by

$$\dim\, A_n = \dim\, SU(n+1) = (n+1)^2 - 1 = n^2 + 2n \tag{230}$$

$$(\#\text{ of roots}) = n^2 + 2n - (n) = n^2 + n\,. \tag{231}$$

The Cartan matrix of A_n is

$$C_{ij} = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & & & ... & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}. \tag{232}$$

Let $\{e_i\}$ with $i = 1, ..., n+1$ be an orthonormal basis such that $(e_i, e_j) = \delta_{ij}$. Then we can construct the roots of A_n as follows

$$\text{fundamental}: \quad \alpha_1 = e_1 - e_2\,, \quad \alpha_2 = e_2 - e_3\,, \quad \ldots\,, \quad \alpha_n = e_n - e_{n+1} \tag{233}$$

$$\text{positive}: \quad e_i - e_j\,, \quad i < j \tag{234}$$

$$\text{negative}: \quad e_i - e_j\,, \quad i > j\,, \tag{235}$$

which correctly amounts to $(n+1)^2 - (n+1) = n^2 + n$ different roots. We can also confirm that we reproduce the Cartan matrix with this choice of basis. The inner products of various simple roots are

$$(\alpha_i, \alpha_i) = (e_i - e_{i+1}, e_i - e_{i+1}) = 2 \tag{236}$$

$$(\alpha_i, \alpha_{i+1}) = (e_i - e_{i+1}, e_{i+1} - e_{i+2}) = -1 \tag{237}$$

$$(\alpha_i, \alpha_{j>i+1}) = 0\,. \tag{238}$$

Therefore, the Cartan matrix elements are

$$C_{i,i+1} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_{i+1})} = 2\frac{-1}{2} = -1\,, \tag{239}$$

with the C_{ii} components being trivially satisfied and all others being equal to zero.

Before we can proceed to the Weyl dimension formula, we want a systematic way of expressing positive roots in terms of the fundamental roots. This can be achieved by noting that for $i < j$

$$\begin{aligned} e_i - e_j &= e_i - e_{i+1} + e_{i+1} - e_{i+2} + \cdots - e_{j-1} + e_{j-1} - e_j \\ &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \\ &= \omega_n C_{ni} + \omega_n C_{n,i+1} + \cdots + \omega_n C_{n,j-1} \end{aligned} \tag{240}$$

where in the last line we have expressed the roots in terms of fundamental weights ω.

The Weyl dimension formula reads

$$\begin{aligned} \dim\,[n_1 ... n_n] &= \prod_{\beta\in\Phi^+} \frac{([n_1+1,\ldots,n_n+1],\beta)}{([1\ldots1],\beta)} \\ &= \boxed{\alpha_1} \ldots \boxed{\alpha_n}\boxed{\alpha_1+\alpha_2} \ldots \boxed{\alpha_{n-1}+\alpha_n} \\ &\quad \times \boxed{\alpha_1+\alpha_2+\alpha_3} \ldots \boxed{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} \ldots \end{aligned} \tag{241}$$

where β is any positive root and the α_i still refer to the fundamental roots. We have split the contributions from different positive roots in terms of their composition of fundamental roots, which we can now inspect on a case-by-case basis. We begin with the simplest

$$\begin{aligned} \boxed{\alpha_1} &= \frac{([n_1+1,\ldots,n_n+1],\alpha_1)}{([1\ldots1],\alpha_1)} \\ &= \frac{(n_1+1)C^{-1}_{1a}(\alpha_a,\alpha_1) + \cdots + (n_n+1)C^{-1}_{na}(\alpha_a,\alpha_1)}{C^{-1}_{1a}(\alpha_a,\alpha_1) + \cdots + C^{-1}_{na}(\alpha_a,\alpha_1)}\,, \end{aligned} \tag{242}$$

where we used

$$\begin{aligned} [n_1+1,\ldots,n_n+1] &= (n_1+1)\omega_1 + \cdots + (n_n+1)\omega_n \\ &= (n_1+1)C^{-1}_{1a}\alpha_a + \cdots + (n_n+1)C^{-1}_{na}\alpha_a\,, \end{aligned} \tag{243}$$

and

$$\begin{aligned} [1,\ldots,1] &= \omega_1 + \cdots + \omega_n \\ &= C^{-1}_{1a}\alpha_a + \cdots + C^{-1}_{na}\alpha_a\,. \end{aligned} \tag{244}$$

Since for all fundamental roots we have that $(\alpha_i, \alpha_i) = 2$, this also means that $(\alpha_i, \alpha_j) = C_{ij}$.

The above simplifies very nicely to

$$\boxed{\alpha_1} = \frac{(n_1+1)C^{-1}_{1a}C_{a1} + \cdots + (n_n+1)C^{-1}_{na}C_{a1}}{C^{-1}_{1a}C_{a1} + \cdots + C^{-1}_{na}C_{a1}} = \frac{n_1+1}{1}\,. \tag{245}$$

This is the bulk of the calculation done, and the other cases follow straightforwardly. Consider an element of the product for a positive root made up of 2 fundamental roots

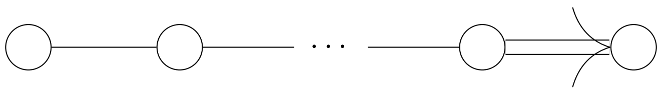
$$\begin{aligned}\boxed{\alpha_1+\alpha_2} &= \frac{([n_1+1,\dots,n_n+1],\alpha_1+\alpha_2)}{([1,\dots,1],\alpha_1+\alpha_2)} \\ &= \frac{([n_1+1,\dots,n_n+1],\alpha_1)+([n_1+1,\dots,n_n+1],\alpha_2)}{([1,\dots,1],\alpha_1)+([1,\dots,1],\alpha_2)} \\ &= \frac{n_1+1+n_2+1}{1+1} = \frac{n_1+n_2+2}{2}\,.\end{aligned} \tag{246}$$

And, therefore, the Weyl dimension formula for a representation of A_n is

$$\begin{aligned}\dim\,[n_1\dots n_n] = (n_1+1)\dots(n_n+1)&\frac{(n_1+n_2+2)}{2}\dots\frac{(n_{n-1}+n_n+2)}{2} \\ &\frac{(n_1+n_2+n_3+3)}{3}\dots\frac{(n_1+n_2+n_3+n_4+4)}{4}\dots\,.\end{aligned} \tag{247}$$

A.2 B_n

Instead of repeating the whole calculation, we will focus on the novel aspects for $B_n \simeq so(2n{+}1)$ and at times not be as rigorous as for the case of A_n. Most fundamentally, the difference is that we now have a short root, as can be seen in the Dynkin diagram



This has to be reflected in our choice of orthonormal basis. As such, let $\{e_i\}$ with $i = 1,\dots,n$ be an orthonormal basis and we define the fundamental roots to be

$$\alpha_1 = e_1 - e_2\,, \quad \dots, \alpha_{n-1} = e_{n-1} - e_n\,, \quad \alpha_n = e_n\,. \tag{248}$$

One can (and should) check that this indeed reproduces the Cartan matrix elements correctly.

A quick calculation shows that there are n^2 positive roots, which we can choose to be

$$e_i\,, \quad e_i - e_j\,, \quad e_i + e_j \quad \text{with} \quad i<j\,. \tag{249}$$

Or, expressed in terms of fundamental roots,

$$e_i = \alpha_i + \cdots + \alpha_n \tag{250}$$

$$e_i - e_j = \alpha_i + \cdots + \alpha_{j-1} \tag{251}$$

$$e_i + e_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n\,. \tag{252}$$

The Weyl dimension formula in terms of block elements is

$$\begin{aligned}\dim\,[n_1 \ldots n_n] &= \prod_{\beta\in\Phi^+} \frac{([n_1+1,\ldots,n_n+1],\beta)}{([1\ldots1],\beta)} \\ &= \left(\boxed{\alpha_1}\ldots\boxed{\alpha_n}\right)\left(\boxed{\alpha_1+\alpha_2}\ldots\boxed{\alpha_{n-1}+\alpha_n}\right)\left(\boxed{\alpha_{n-1}+2\alpha_n}\right) \\ &\quad \left(\boxed{\alpha_1+\alpha_2+\alpha_3}\ldots\right)\left(\boxed{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n}\,\boxed{\alpha_{n-2}+2\alpha_{n-1}+2\alpha_n}\right)\ldots\,.\end{aligned} \tag{253}$$

For the blocks with a single fundamental root, all except the last one follow in exactly the same way as for A_n. The last one is

$$\begin{aligned}\boxed{\alpha_n} &= \frac{([n_1+1,\ldots,n_n+1],\alpha_n)}{([1,\ldots,1],\alpha_n)} \\ &= \frac{\cdots + (n_n+1)C^{-1}_{na}(\alpha_a,\alpha_n)}{\cdots + C^{-1}_{na}(\alpha_a,\alpha_n)},\end{aligned} \tag{254}$$

in a similar manner as before, we note that $(\alpha_a,\alpha_n) = \frac{1}{2}(\alpha_n,\alpha_n)C_{an} = \frac{1}{2}C_{an}$, and hence

$$\boxed{\alpha_n} = \frac{(n_n+1)\frac{1}{2}}{\frac{1}{2}} = n_n + 1\,. \tag{255}$$

And for other boxed elements that include the last root, we find

$$\begin{aligned}\boxed{\alpha_{n-1}+\alpha_n} &= \frac{n_{n-1}+1+\frac{1}{2}(n_n+1)}{1+\frac{1}{2}} \\ &= \frac{2n_{n-1}+n_n+3}{3}\end{aligned} \tag{256}$$

$$\begin{aligned}\boxed{\alpha_{n-1}+2\alpha_n} &= \frac{n_{n-1}+1+\frac{2}{2}(n_n+1)}{1+\frac{2}{2}} \\ &= \frac{n_{n-1}+n_n+2}{2}\,.\end{aligned} \tag{257}$$

And, finally, we have the Weyl dimension formula for representations of B_n

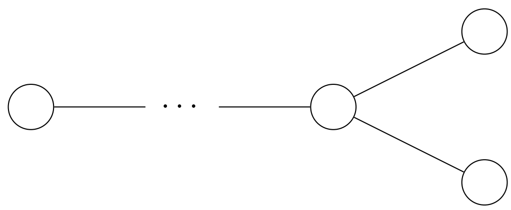
$$\begin{aligned}\dim\,[n_1 \dots n_n] = &\Big((n_1+1)\dots(n_n+1)\Big)\left(\frac{n_1+n_2+2}{2}\dots\frac{n_{n-1}+n_n+2}{2}\right)\\ &\left(\frac{2n_{n-1}+n_n+3}{3}\right)\left(\frac{n_1+n_2+n_3+3}{3}\dots\frac{n_{n-2}+n_{n-1}+n_n+3}{3}\right)\\ &\left(\frac{n_{n-2}+2n_{n-1}+n_n+4}{4}\,\frac{2n_{n-2}+2n_{n-1}+n_n+5}{5}\right)\dots\end{aligned} \tag{258}$$

where we have slightly reshuffled the order of the roots such that it takes a nicer form.

A.3 C_n

A.4 D_n

Again, the root structure of $D_n \simeq so(2n)$ is best encapsulated in the Dynkin diagram



and the corresponding Cartan matrix

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & & & \dots & & & & \\ & & & & -1 & 2 & -1 & -1 \\ & & & & & -1 & 2 & 0 \\ & & & & & -1 & 0 & 2 \end{pmatrix}. \tag{259}$$

We can express the fundamental roots in terms of an orthonormal basis $\{e_i\}$ with $i = 1, \dots, n$ as

$$\alpha_1 = e_1 - e_2\,, \quad \dots\,, \quad \alpha_{n-1} = e_{n-1} - e_n\,, \quad \alpha_n = e_{n-1} + e_n\,. \tag{260}$$

The $n^2 - n$ positive roots can be expressed as

$$e_i - e_j\,, \quad e_i + e_j \quad \text{with} \quad i < j\,. \tag{261}$$

The first set of these is easily expressed for all $i < j$ in terms of fundamental roots as

$$e_i - e_j = \alpha_i + \cdots + \alpha_{j-1} \,. \tag{262}$$

For the second set, we split it up into the cases before we reach the fork in the Dynking diagram and the ones including the fork

$$e_i + e_j = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \quad \text{for} \quad j \leq n-2 \tag{263}$$

$$e_i + e_{n-1} = \alpha_i + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n \tag{264}$$

$$e_i + e_n = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n \tag{265}$$

$$e_{n-1} + e_n = \alpha_n \,. \tag{266}$$

The Weyl dimension formula can then be written as

$$\begin{aligned}\dim\,[n_1 \ldots n_n] &= \prod_{\beta \in \Phi^+} \frac{([n_1+1, \ldots, n_n+1], \beta)}{([1 \ldots 1], \beta)} \\ &= \left(\boxed{\alpha_1} \ldots \boxed{\alpha_n}\right) \left(\boxed{\alpha_1 + \alpha_2} \ldots \boxed{\alpha_{n-2} + \alpha_{n-1}}\right) \left(\boxed{\alpha_{n-1} + \alpha_n}\right) \left(\boxed{\alpha_{n-2} + \alpha_n}\right) \\ &\quad \left(\boxed{\alpha_1 + \alpha_2 + \alpha_3} \ldots \boxed{\alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}}\right) \left(\boxed{\alpha_{n-2} + \alpha_{n-1} + \alpha_n}\right) \\ &\quad \left(\boxed{\alpha_{n-3} + \alpha_{n-2} + \alpha_n}\right) \ldots \,.\end{aligned} \tag{267}$$

where we have put terms in separate brackets if they belong to a different case. Since they are all the same size, we can use the results from A.1, and the dimension formula for D_n becomes

$$\begin{aligned}\dim\,[n_1 \ldots n_n] = (n_1+1) \ldots (n_n+1) &\left(\frac{n_1+n_2+2}{2} \ldots \frac{n_{n-1}+n_n+2}{2}\right) \left(\frac{n_{n-2}+n_n+2}{2}\right) \\ &\left(\frac{n_1+n_2+n_3+3}{3} \ldots \frac{n_{n-2}+n_{n-1}+n_n+3}{3}\right) \left(\frac{n_{n-3}+n_{n-2}+n_n+3}{3}\right) \ldots\end{aligned} \tag{268}$$

where we have combined two of the grouped brackets together as they have the same form.

References